

Characteristic cycle and the Euler number of a constructible sheaf on a surface *

TAKESHI SAITO

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Abstract

We define the characteristic cycle of a constructible sheaf on a smooth surface in the cotangent bundle. We prove that the intersection number with the 0-section equals the Euler number and that the total dimension of vanishing cycles at an isolated characteristic point is also computed as an intersection number.

For a constructible sheaf on a smooth algebraic variety in positive characteristic, an analogy between the wild ramification of an étale sheaf and the irregularity of a \mathcal{D} -module in characteristic 0 suggests that the characteristic cycle is defined as a cycle of the cotangent bundle. Its intersection product with the 0-section is expected to give the characteristic class [4] and the Euler number consequently. At an isolated characteristic point (see the last paragraph of Section 1 for the definition) of a fibration to a curve, the intersection number with the section defined by a non-vanishing differential form of the curve is expected to be equal to the total dimension of nearby cycles.

In a tamely ramified case, the characteristic cycle has an elementary definition in arbitrary dimension (1.4). For a sheaf on a curve, the characteristic cycle is determined by the Swan conductor at the boundary (1.6). For a sheaf on a surface, Deligne and Laumon define the characteristic cycle implicitly in [20] (see also [16, Letter 3 (b)]) using the total dimension of the nearby cycles and compute the Euler number, under the “non-feroce” assumption.

To remove the assumption, Deligne further sketched a global method, extending that in [6], in a letter [8] and in unpublished notes [9] with more detail. The method fits in an approach of Beilinson using the Radon transform [5].

In this article, we define the characteristic cycle of a sheaf on a surface in general in Definition 3.8, by combining the approach using the Radon transform and the ramification theory developed in [24], following the ideas in [8] and [9]. We prove that the intersection number with the 0-section equals the Euler number in general in Theorem 3.18 and that that with the section defined by a fibration to a curve computes the total number of nearby cycles at an isolated characteristic point in Theorem 3.16. We also show in Proposition 3.19 that it is the same as that defined in [24, Definition 3.5] as long as the latter is defined. The relation with the characteristic class defined in [4] is still to be clarified.

The definition goes as follows. First, by studying the ramification of the Radon transform using the ramification theory developed in [24], we define the characteristic cycle

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that a priori may depend on the choice of a projective embedding. Using a deformation argument [9] and the dimension formula for the nearby cycles by Deligne and Laumon [19], we show that the characteristic cycle thus defined computes the total dimension of nearby cycles at an isolated characteristic point. We deduce from this that the characteristic cycle is in fact independent of the choice of a projective embedding.

The deformation argument relies on the stability of nearby cycles under small deformation of fibrations. This in turn follows from a generalization of Hensel's lemma due to Elkik [10] together with the vanishing of the limit of nearby cycles for a certain sequence of blow-up and the stability of the dimension of nearby cycles. The last fact is based on a generalization by Kato [17], [14] of the formula [19] used above and the stability of the ramification of restrictions to curves.

We prove that the Euler number equals the intersection number of the characteristic cycle with the 0-section, applying the Grothendieck-Ogg-Shafarevich formula computing the Euler number of a sheaf on a curve [13] two times. The equality implies that the difference with the characteristic cycle defined in [24] is controlled by a divisor numerically equivalent to zero. By using a finite covering trivializing the ramification except at one irreducible component of the ramification divisor, we conclude that this divisor is in fact zero and derive the coincidence of the two definitions.

We briefly describe the content of each section. After briefly recalling the ramification theory developed in [24] in Section 1, we prove in Section 2.1 the stability of the ramification of the restrictions to curves in Propositions 2.1 and 2.3. We also show a continuity of the total dimension of nearby cycles in Proposition 2.5. Using a generalization of Hensel's lemma due to Elkik [10] recalled in Section 2.2, we prove the stability Theorem 2.13 of nearby cycles under small deformation of fibrations in Section 2.3.

After some preliminaries on the universal family of hyperplane sections in Section 3.1, we study the ramification of the Radon transform and define the characteristic cycle in Definition 3.8, depending on projective embedding in Section 3.2. We prove in Proposition 3.12 and Theorem 3.16 a formula computing the total dimension of nearby cycles as an intersection number with the characteristic cycle and deduce that it is in fact independent of a projective embedding. Finally in Section 3.3, we prove the equality for the Euler number in Theorem 3.18 and the equality of the characteristic cycle defined using the Radon transform with that defined in [24] in Proposition 3.19.

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1 Brief summary of ramification theory

We briefly recall the ramification theory from [24]. Let K be a complete discrete valuation field with not necessarily perfect residue field F of characteristic $p > 0$. The filtration $(G_K^r)_r$ by (non-logarithmic) ramification groups of the absolute Galois group $G_K = \text{Gal}(K^{\text{sep}}/K)$ is defined as a decreasing filtration by closed normal subgroups indexed by rational numbers $r \geq 1$ [2], [3]. For a rational number $r \geq 1$, define $G_K^{r+} \subset G_K^r$ to be the closure $\overline{\bigcup_{s>r} G_K^s}$. The subgroup $I_K = G_K^1 \subset G_K$ is the inertia subgroup and $P_K = G_K^{1+} \subset G_K^1$ is its p -Sylow subgroup called the wild inertia subgroup. Assume that K is of characteristic p . Then, the graded piece $\text{Gr}^r G_K = G_K^r / G_K^{r+}$ is known to be an abelian group annihilated by p [24, Corollary 2.28.1] for $r > 1$.

Let Λ be a finite field of characteristic $\ell \neq p$ and M be a finite Λ -module with continuous G_K -action. Then, there exists a unique decomposition $M = \bigoplus_{r \geq 1} M^{(r)}$ called the slope decomposition characterized by the condition that the G_K^{r+} -fixed part $M^{G_K^{r+}}$ equals $\bigoplus_{s \leq r} M^{(s)}$ for $r \geq 1$. We define the total dimension of M by

$$(1.1) \quad \dim \text{tot}_K M = \sum_{r \geq 1} r \cdot \dim_{\Lambda} M^{(r)}.$$

In the classical case where the residue field F is perfect, it is equal to the sum $\dim M + \text{Sw}_K M$ of the dimension and the Swan conductor [13, Section 4]. Further, if K is of characteristic p and if Λ contains a primitive p -th root of unity, the r -th piece $M^{(r)}$ for $r > 1$ is decomposed as $\bigoplus_{\chi: \text{Gr}^r G_K \rightarrow \Lambda^\times} \chi^{\oplus n(\chi)}$ by characters of the abelian group $\text{Gr}^r G_K$ annihilated by p .

We consider the case where X is a smooth scheme over a perfect field k of characteristic $p > 0$ and $K = \text{Frac}(\hat{\mathcal{O}}_{X,\xi})$ is the local field at the generic point ξ of a smooth irreducible divisor D . The residue field F is the function field $\kappa(\xi)$ of the divisor D and the residue field \bar{F} of K^{sep} is an algebraic closure of F . For a rational number r , let $I(r)$ denote the fractional ideal $\{a \in K^{\text{sep}} \mid \text{ord}_K a \geq -r\}$ and define an \bar{F} -vector space $L(r) = I(r) \otimes_{\mathcal{O}_{K^{\text{sep}}}} \bar{F}$ of dimension 1. Then, the dual $(\text{Gr}^r G_K)^\vee = \text{Hom}_{\mathbf{F}_p}(\text{Gr}^r G_K, \mathbf{F}_p)$ of the \mathbf{F}_p -vector space $\text{Gr}^r G_K$ is canonically identified as a subgroup of the \bar{F} -vector space $\Omega_{X/k,\xi}^1 \otimes_{\mathcal{O}_{X,\xi}} L(r)$ by the canonical injection

$$(1.2) \quad \text{char}: (\text{Gr}^r G_K)^\vee \rightarrow \Omega_{X/k,\xi}^1 \otimes_{\mathcal{O}_{X,\xi}} L(r)$$

defined in [24, Corollary 2.28.2]. For a non-trivial character $\chi \in (\text{Gr}^r G_K)^\vee$, let $F(\chi)$ denote a finite extension of F where $\text{char}(\chi)$ regarded as an \bar{F} -linear mapping $L(-r) \rightarrow \Omega_{X/k,\xi}^1 \otimes_{\mathcal{O}_{X,\xi}} \bar{F}$ descends to an $F(\chi)$ -linear mapping. Then, it defines a line $L(\chi)$ in the fiber $T^*X \times_X \text{Spec } F(\chi)$ of the cotangent bundle $T^*X = \mathbf{V}(\Omega_{X/k}^1)$ at $\text{Spec } F(\chi) \rightarrow \xi \in X$.

Let $U = X - D$ denote the complement and $j: U \rightarrow X$ be the open immersion. Let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on U . We assume that Λ contains a primitive p -th root of unity, fix an isomorphism $\mathbf{F}_p \rightarrow \mu_p(\Lambda)$ and identify $(\mathrm{Gr}^r G_K)^\vee = \mathrm{Hom}(\mathrm{Gr}^r G_K, \Lambda^\times)$. Then, the characteristic cycle $\mathrm{Char} \, j_! \mathcal{F}$ is defined on a neighborhood of the generic point ξ of D as follows. Let $\bar{\eta}$ be the geometric point of U defined by the separable closure K^{sep} and let $M = \mathcal{F}_{\bar{\eta}}$ be the continuous representation of G_K defined by \mathcal{F} . Then the slope decomposition $M = \bigoplus_{r \geq 1} M^{(r)}$ and the decomposition by characters $M^{(r)} = \bigoplus_{\chi \in (\mathrm{Gr}^r G_K)^\vee} \chi^{\oplus n(\chi)}$ for $r > 1$ are defined. Let $T_X^* X \subset T^* X$ denote the 0-section and $T_D^* X \subset T^* X$ the conormal bundle. We define the germ of the characteristic cycle $\mathrm{Char} \, j_! \mathcal{F}$ at ξ to be

$$(1.3) \quad (-1)^d \left(\mathrm{rank} \, \mathcal{F} \cdot [T_X^* X] + \dim M^{(1)} \cdot [T_D^* X] + \sum_{r > 1} r \cdot \sum_{\chi \in (\mathrm{Gr}^r G_K)^\vee} \frac{n(\chi)}{[F(\chi): F]} [L(\chi)] \right).$$

Let $(\mathrm{Char} \, j_! \mathcal{F})_D^{\mathrm{wild}}$ denote the sum of the last term in the parentheses.

More generally, we consider the case where D is not necessarily an irreducible and smooth divisor. After removing closed subset of codimension ≥ 2 if necessary, we assume that D is a divisor with simple normal crossings. Let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on $U = X - D$. Then, farther after removing closed subset of codimension ≥ 2 if necessary, we may assume that the ramification of \mathcal{F} along D is non-degenerate in the sense of [24, Definition 3.1] which we will not recall here.

Assuming that the ramification of \mathcal{F} along D is non-degenerate, the characteristic cycle $\mathrm{Char} \, j_! \mathcal{F}$ is defined as follows. Let D_1, \dots, D_m be the irreducible components of D and for a subset $I \subset \{1, \dots, m\}$, let D_I denote the intersection $\bigcap_{i \in I} D_i$ and $T_{D_I}^* X \subset T^* X$ be the conormal bundle. If \mathcal{F} is tamely ramified along D , the characteristic cycle $\mathrm{Char} \, j_! \mathcal{F}$ is defined by

$$(1.4) \quad \mathrm{Char} \, j_! \mathcal{F} = (-1)^d \left(\sum_{I \subset \{1, \dots, m\}} \mathrm{rank} \, \mathcal{F} \cdot [T_{D_I}^* X] \right).$$

Next, we consider the case where the ramification of \mathcal{F} along D is non-degenerate and totally wild; for every irreducible component D_i of D , the tame part $\mathcal{F}_{\bar{\eta}_i}^{(1)}$ is 0. Then the germ of cycle $(\mathrm{Char} \, j_! \mathcal{F})_{D_i}^{\mathrm{wild}}$ in (1.3) for each irreducible component D_i of D is defined as a cycle of $T^* X$ and the characteristic cycle is defined by the equality

$$(1.5) \quad \mathrm{Char} \, j_! \mathcal{F} = (-1)^d \left(\mathrm{rank} \, \mathcal{F} \cdot [T^* X] + \sum_i (\mathrm{Char} \, j_! \mathcal{F})_{D_i}^{\mathrm{wild}} \right).$$

In general, we define $\mathrm{Char} \, j_! \mathcal{F}$ by additivity and étale descent. Define the singular support $SS(j_! \mathcal{F}) \subset T^* X$ to be the union of the underlying set of the components of the characteristic cycle $\mathrm{Char} \, j_! \mathcal{F}$. If $\dim X = 1$, we have

$$(1.6) \quad \mathrm{Char} \, j_! \mathcal{F} = - \left(\mathrm{rank} \, \mathcal{F} \cdot [T_X^* X] + \sum_{x \in D} (\mathrm{rank} \, \mathcal{F} + \mathrm{Sw}_x \mathcal{F}) \cdot [T_x^* X] \right).$$

Going back to the general dimension, the total dimension divisor is defined by

$$(1.7) \quad DT j_! \mathcal{F} = \sum_i \dim \mathrm{tot}_{K_i} \mathcal{F}_{\bar{\eta}_i} \cdot D_i$$

where the geometric point $\bar{\eta}_i$ is defined by a separable closure of the local field K_i at the generic point of an irreducible component D_i of D . Note that in the definition of the total dimension divisor we do not need to assume that ramification is non-degenerate.

More generally, for a constructible complex \mathcal{K} of Λ -module on X such that the restriction of the cohomology sheaf $\mathcal{H}^q \mathcal{K}$ is locally constant on U for every integer q , the Artin divisor $a(\mathcal{K})$ is defined by

$$(1.8) \quad a(\mathcal{K}) = \sum_q (-1)^q \left(DT j_{!} j^* \mathcal{H}^q(\mathcal{K}) - \sum_i \dim \mathcal{H}^q(\mathcal{K})_{\bar{\xi}_i} \cdot D_i \right)$$

where $\bar{\xi}_i$ is a geometric point dominating the generic point of an irreducible component D_i of D .

Let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on $U = X - D$ with non-degenerate ramification along a divisor D with simple normal crossings. Let C be a smooth curve over k and $C \rightarrow X$ be an immersion over k . We say that the immersion $C \rightarrow X$ is non-characteristic at x with respect to $j_{!} \mathcal{F}$ if the tangent vector of C at x is not annihilated by any nonzero differential form in the fiber of $SS(j_{!} \mathcal{F})$ at x . If $C \rightarrow X$ is non-characteristic at x , the total dimension divisor is compatible with the pull-back [24, Proposition 3.8]:

$$(1.9) \quad (DT j_{!} \mathcal{F}, C)_x = \dim \text{tot}_x \mathcal{F}|_C.$$

Let C be a smooth curve over k and $f: X \rightarrow C$ be a smooth morphism over k . We say that $f: X \rightarrow C$ is *non-characteristic* with respect to $j_{!} \mathcal{F}$ if the section of T^*X defined by the pull-back by f of a non-vanishing differential form on C does not intersect with the singular support $SS(j_{!} \mathcal{F})$. We say that $x \in X$ is a characteristic point of $f: X \rightarrow C$ with respect to $j_{!} \mathcal{F}$ if $f: X \rightarrow C$ is not non-characteristic on a neighborhood of x . A morphism $f: X \rightarrow C$ non-characteristic with respect to $j_{!} \mathcal{F}$ is universally locally acyclic relatively to $j_{!} \mathcal{F}$, if either \mathcal{F} is tamely ramified along D or \mathcal{F} is totally wildly ramified along D and $f|_D: D \rightarrow C$ is flat by [24, Proposition 3.15]. In particular, the complex of vanishing cycles $\phi(j_{!} \mathcal{F}, f)$ on the geometric fiber $X_{\bar{c}}$ is 0 for every geometric closed point \bar{c} of C .

We say that a closed point x is an isolated characteristic point of $f: X \rightarrow C$ with respect to $j_{!} \mathcal{F}$ if the restriction of f to a neighborhood of x is non-characteristic with respect to $j_{!} \mathcal{F}$ except possibly at x . This definition makes sense also in the case if only $X - \{x\}$ is assumed smooth over k .

2 Stability of nearby cycles

2.1 Stability of ramification of the restrictions to curves

For morphisms $f: X \rightarrow S$ and $T \rightarrow S$ of schemes, let $(X, f) \times_S T$ denote the fibered product to indicate the morphism, if necessary. For morphisms $f: X \rightarrow S$ and $g: X \rightarrow S$ of schemes and closed subscheme Z of X defined by the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, we say f and g are congruent to each other modulo \mathcal{I}_Z and write $f \equiv g \pmod{\mathcal{I}_Z}$ if the restrictions $f|_Z: Z \rightarrow S$ and $g|_Z: Z \rightarrow S$ are the same.

Proposition 2.1. *Let X be a normal surface over a perfect field k of characteristic $p > 0$ and let $f: X \rightarrow C$ be a flat morphism over k to a smooth curve C over k . Let Λ be a finite field of characteristic $\ell \neq p$ and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on the complement $U = X - D$ of a closed subscheme $D \subset X$ quasi-finite over C . Let u be a closed point of D such that u is an isolated characteristic point of $f: X \rightarrow C$ with respect to $j_!\mathcal{F}$.*

1. *There exists an integer $N \geq 1$ such that if a morphism $g: X \rightarrow C$ over k satisfies $g \equiv f \pmod{\mathfrak{m}_u^N}$, then its restriction $g|_D: D \rightarrow C$ is quasi-finite at u and that u is an isolated characteristic point of $g: X \rightarrow C$ with respect to $j_!\mathcal{F}$. Further, if $f|_D: D \rightarrow C$ is flat at u (resp. and if $f|_{D - \{u\}}: D - \{u\} \rightarrow C$ is étale), then we require $g|_D: D \rightarrow C$ is flat at u (resp. and $g|_{D - \{u\}}: D - \{u\} \rightarrow C$ is étale on a neighborhood of u except at u).*

2. *There exists an integer $N \geq 1$ such that if a morphism $g: X \rightarrow C$ over k satisfies $g \equiv f \pmod{\mathfrak{m}_u^N}$, there exist an étale neighborhood $V \rightarrow C$ of $v = f(u)$ such that the connected components $(D, f|_D)_V^0$ and $(D, g|_D)_V^0$ of $(D, f|_D) \times_C V$ and $(D, g|_D) \times_C V$ containing u are finite over V and that for every closed point $y \in V - \{v\}$, we have*

$$(2.1) \quad \sum_{x \in (D, f|_D)_V^0, f(x)=y} \dim \text{tot}_x(\mathcal{F}|_{f^{-1}(y)}) = \sum_{x \in (D, g|_D)_V^0, g(x)=y} \dim \text{tot}_x(\mathcal{F}|_{g^{-1}(y)}).$$

Proof. 1. Let $N \geq 2$ be an integer such that \mathfrak{m}_u^{N-1} annihilates $(D, f|_D) \times_C v$ where $v = f(u)$. If $g \equiv f \pmod{\mathfrak{m}_u^N}$, then \mathfrak{m}_u^{N-1} also annihilates $(D, g|_D) \times_C v$ and hence $g|_C: D \rightarrow C$ is quasi-finite at u .

Assume $f|_D: D \rightarrow C$ is flat at u . Then, the pull-back by f of a uniformizer $t \in \mathcal{O}_{C,v}$ forms a regular sequence of the local ring $\mathcal{O}_{D,u}$ and so is the pull-back by g . Hence $g|_D: D \rightarrow C$ is also flat at u . Further if $f|_D: D \rightarrow C$ is étale except at u , let $N \geq 2$ be an integer such that \mathfrak{m}_u^{N-1} annihilates $\Omega_{D/C,u}$ with respect to $f|_D$. If $g \equiv f \pmod{\mathfrak{m}_u^N}$, then \mathfrak{m}_u^{N-1} also annihilates $\Omega_{D/C,u}$ with respect to $f|_D$ and hence $g|_D: D \rightarrow C$ is étale on a neighborhood of u except at u .

Let $\pi: X' \rightarrow X$ be a resolution. Namely, X' is a smooth surface over k , π is proper and $X' - \pi^{-1}(u) \rightarrow X - \{u\}$ is an isomorphism. The singular support $SS(j_!\mathcal{F})$ is defined as a closed subset of $T^*(X - \{u\})$. Let $SS(j_!\mathcal{F})' \subset T^*X'$ denote the closure of $SS(j_!\mathcal{F})$ and regard it as a reduced closed subscheme. Let $E \subset X'$ denote the inverse image $\pi^{-1}(u) = X' \times_X u$.

Let $t \in \mathcal{O}_{C,v}$ be a uniformizer and let $df: X' \rightarrow T^*X'$ denote the section defined by f^*dt on a neighborhood of E . By the assumption that u is an isolated characteristic point of $f: X \rightarrow C$ with respect to $j_!\mathcal{F}$, the intersection $(X', df) \times_{T^*X'} SS(j_!\mathcal{F})'$ of the image of the section df and the singular support $SS(j_!\mathcal{F})'$ is a subset of the inverse image $T^*X' \times_{X'} E$. Let $N \geq 2$ be an integer such that $(X', df) \times_{T^*X'} SS(j_!\mathcal{F})'$ is a closed subscheme annihilated by \mathcal{I}_E^{N-2} . Since $g \equiv f \pmod{\mathfrak{m}_u^N}$ implies $dg \equiv df \pmod{\mathfrak{m}_u^{N-1}\Omega_{X/k}^1}$, the intersection $(X', dg) \times_{T^*X'} SS(j_!\mathcal{F})'$ is also annihilated by \mathcal{I}_E^{N-2} on a neighborhood of $T^*X' \times_{X'} E$ and u is an isolated characteristic point of $g: X \rightarrow C$ with respect to $j_!\mathcal{F}$.

2. Shrinking X if necessary, we may assume that u is the unique point in the fiber of $f|_D: D \rightarrow C$. Since a quasi-finite scheme over a henselian discrete valuation ring is the disjoint union of a finite scheme and a flat scheme, there exist an étale neighborhood of $V \rightarrow C$ of $v = f(u)$ such that the connected components $(D, f|_D)_V^0$ and $(D, g|_D)_V^0$ are finite. It suffices to consider the case where they are finite and flat.

Let $DT(j_!\mathcal{F}, f)_V^0$ denote the part of the pull-back of $DT(j_!\mathcal{F})$ to $(X, f)_V = (X, f) \times_C V$ supported on $(D, f|_D)_V^0$ and similarly for $DT(j_!\mathcal{F}, g)_V^0$. Since u is an isolated characteristic

point of $f: X \rightarrow C$ with respect to $j_! \mathcal{F}$, the left hand side of (2.1) is equal to the degree of $DT(j_! \mathcal{F}, f)_V^0$ over V by (1.9). We will take an integer $N \geq 1$ satisfying the conditions in 1. Then, for a morphism $g: X \rightarrow C$ over k satisfying $g \equiv f \pmod{\mathfrak{m}_u^N}$, the point u is an isolated characteristic point of $g: X \rightarrow C$ with respect to $j_! \mathcal{F}$ and the right hand side is also equal to the degree of $DT(j_! \mathcal{F}, g)_V^0$ over V .

Let $N \geq 1$ be an integer such that $DT(j_! \mathcal{F}, f)_V^0 \times_V v$ is annihilated by \mathfrak{m}_u^{N-1} . If $g \equiv f \pmod{\mathfrak{m}_u^N}$, then $DT(j_! \mathcal{F}, g)_V^0 \times_V v$ is equal to $DT(j_! \mathcal{F}, f)_V^0 \times_V v$ and is also annihilated by \mathfrak{m}_u^{N-1} . Thus the degree of $DT(j_! \mathcal{F}, f)_V^0$ over V is equal to that of $DT(j_! \mathcal{F}, g)_V^0$ and the assertion follows. \square

The following example shows that in Proposition 2.1 and Theorem 2.13, one cannot drop the assumption of non-characteristicity.

Example 2.2. Let $X = \mathbf{A}^2 = \text{Spec } k[x, y]$ be the affine plane over an algebraically closed field k of characteristic $p > 2$ and $U = X - D$ be the complement of the y -axis D . Let $u = (0, 0)$ denote the origin of $X = \mathbf{A}^2$. Assume that Λ contains a primitive p -th root of unity and let \mathcal{F} be the locally constant constructible sheaf of Λ -modules of rank 1 on U defined by the Artin-Schreier equation $z^p - z = \frac{y}{x^p}$. Then, the singular support $SS(j_! \mathcal{F})$ is the union of the zero-section $T_X^* X$ and the sub line bundle over D spanned by the section dy .

Let $f: X \rightarrow C = \mathbf{A}^1 = \text{Spec } k[t]$ be the smooth morphism defined by $t \mapsto y$. It is characteristic with respect to $j_! \mathcal{F}$ at every point of D . The restriction $f|_D: D \rightarrow C$ is an isomorphism. For $c \in \mathbf{A}^1(k)$, the Swan conductor $\text{Sw}_{(0,c)}(j_! \mathcal{F}|_{f^{-1}(c)})$ of the restriction to the fiber is 0 for $c = 0$ and 1 for $c \neq 0$. Hence by [19], we have $\dim \phi_u^1(j_! \mathcal{F}, f) = 1$.

Let $n \geq 2$ be an integer and $g: X \rightarrow C = \mathbf{A}^1 = \text{Spec } k[t]$ be the smooth morphism defined by $t \mapsto y + xy^n$. We have $g \equiv f \pmod{\mathfrak{m}_u^{n+1}}$. The restriction $g|_D: D \rightarrow C$ is also an isomorphism. For $c \in \mathbf{A}^1(k)$, the Swan conductor $\text{Sw}_{(0,c)}(j_! \mathcal{F}|_{g^{-1}(c)})$ of the restriction to the fiber is 0 for $c = 0$ and $p-1 > 1$ for $c \neq 0$. Hence by [19], we have $\dim \phi_u^1(j_! \mathcal{F}, g) = p-1 > 1$.

For closed subschemes C and C' and a closed subscheme Z of X defined by the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, we say $C \equiv C' \pmod{\mathcal{I}_Z}$ if $C \times_X Z = C' \times_X Z$. If $f: X \rightarrow S$ and $g: X \rightarrow S$ satisfy $f \equiv g \pmod{\mathcal{I}_Z}$ and $T \subset S$ is closed subscheme, we have $(X, f) \times_S T \equiv (X, g) \times_S T \pmod{\mathcal{I}_Z}$.

Let C be a reduced excellent noetherian scheme of dimension 1 and u be a closed point of C with perfect residue field. Let $C' \rightarrow C$ be the normalization. Let Λ be a finite field of characteristic ℓ invertible at u and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on $U = C - \{u\}$. Then, total dimension $\dim \text{tot}_u \mathcal{F}$ is defined as the sum $\sum_{u' \in C' \times_C u} \dim \text{tot}_{u'} \mathcal{F}$.

Proposition 2.3. Let X be a normal excellent noetherian scheme of dimension 2 and u be a closed point of X such that $\mathcal{O}_{X,u}$ is of dimension 2 and that the residue field is perfect. Let $U \subset X$ be a dense open subscheme. Let Λ be a finite field of characteristic ℓ invertible on X and let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on U .

Let C be a reduced Cartier divisor of X containing u such that u is in the closure of $C \cap U$. Then, there exists an integer $N \geq 1$ such that for a reduced Cartier divisor C_1 of X satisfying $C \equiv C_1 \pmod{\mathfrak{m}_u^N}$, the point u is in the closure of $C_1 \cap U$ and we have

$$(2.2) \quad \dim \text{tot}_u(\mathcal{F}|_{C \cap U}) = \dim \text{tot}_u(\mathcal{F}|_{C_1 \cap U}).$$

Proof. Let Z be a closed subscheme of X such that $U = X - Z$ and let $N \geq 2$ be an integer such that $\mathcal{O}_{Z \cap C, u}$ is annihilated by \mathfrak{m}_u^{N-1} . Then, for a reduced Cartier divisor C_1 of X of dimension 1 satisfying $C \equiv C_1 \pmod{\mathfrak{m}_u^N}$, the point u is contained in the closure of $C_1 \cap U$.

Let $V \rightarrow U$ be a G -torsor for a finite group G such that the pull-back \mathcal{F}_V of \mathcal{F} is constant and let $f: Y \rightarrow X$ be the normalization of X in V . Let \bar{D} and \bar{D}_1 be the normalizations of $D = C \times_X Y$ and $D_1 = C_1 \times_X Y$. For $\sigma \in G, \neq 1$ and a point v of \bar{D} above u , let $\bar{\mathcal{I}}_{\sigma, v}$ denote the ideal of $\mathcal{O}_{\bar{D}, v}$ defining the intersection $\Delta_{\bar{D}} \cap \Gamma_{\sigma} \subset \Delta_{\bar{D}} = \bar{D}$ of the diagonal and the graph of σ in $\bar{D} \times_C \bar{D}$ and similarly $\bar{\mathcal{I}}_{\sigma, v_1}$ for v_1 of \bar{D}_1 above u . By the definition of the Swan conductor, it suffices to show the existence of $N \geq 1$ such that the congruence $C \equiv C_1 \pmod{\mathfrak{m}_u^N}$ implies a bijection $\bar{D} \times_C \{u\}$ and $\bar{D}_1 \times_C \{u\}$ satisfying the equalities $\text{length } \mathcal{O}_{\bar{D}, v} / \bar{\mathcal{I}}_{\sigma, v} = \text{length } \mathcal{O}_{\bar{D}_1, v_1} / \bar{\mathcal{I}}_{\sigma, v_1}$ for the corresponding points and for $\sigma \in G, \neq 1$.

First, we prove the case where X, C and $D = C \times_X Y$ are regular. For $\sigma \in G, \neq 1$, let \mathcal{I}_{σ} denote the ideal of \mathcal{O}_Y defining the intersection $Y_{\sigma} = \Delta_Y \cap \Gamma_{\sigma} \subset \Delta_Y = Y$ of the diagonal and the graph of σ in $Y \times_X Y$. We have $\bar{\mathcal{I}}_{\sigma, v} = \mathcal{I}_{\sigma} \mathcal{O}_{D, v}$. Let $N \geq 2$ be an integer such that $\mathcal{O}_{D, v} / \mathcal{I}_{\sigma} \mathcal{O}_{D, v}$ is annihilated by \mathfrak{m}_u^{N-1} for every $\sigma \neq 1$ and $v \in f^{-1}(u)$. Let C_1 be an integral closed subscheme of dimension 1 satisfying $C \equiv C_1 \pmod{\mathfrak{m}_u^N}$. Then, since $D_1 = C_1 \times_X Y \equiv D \pmod{\mathfrak{m}_u^2}$, the scheme D_1 is also regular at every $v \in f^{-1}(u)$. Further, $\mathcal{O}_{D_1, v} / \mathcal{I}_{\sigma} \mathcal{O}_{D_1, v}$ is annihilated by \mathfrak{m}_u^{N-1} and is isomorphic to $\mathcal{O}_{D, v} / \mathcal{I}_{\sigma} \mathcal{O}_{D, v}$ for every $\sigma \neq 1$ and $v \in f^{-1}(u)$. Thus the assertion is proved in this case.

We show the general case by reducing to the case proved above by using the following embedded resolution.

Lemma 2.4 (cf. [22, Theorems 8.3.4, 9.2.26]). *Let X be a normal excellent noetherian scheme of dimension 2 and $C \subset X$ be a reduced closed subscheme of dimension 1. Let $U \subset X$ be the complement of finitely many closed points of codimension 2 of X contained in C such that U and $C \cap U$ are regular.*

Then, there exist a regular excellent noetherian scheme X' of dimension 2 and a proper morphism $g: X' \rightarrow X$ such that $g^{-1}(U) \rightarrow U$ is an isomorphism and that the reduced part of the inverse image $g^{-1}(C)$ is a divisor of X' with simple normal crossings.

In particular, the closure $C' \subset X'$ of $g^{-1}(C \cap U)$ with the reduced scheme structure is regular and meets transversely the reduced part E of the effective Cartier divisor $C \times_X X' - C'$.

By shrinking X if necessary, we may assume that $X - \{u\}$ and $C - \{u\}$ are regular. We apply Lemma 2.4 to $X - \{u\}$ to obtain $g: X' \rightarrow X$ and further to the inverse image $D' = C' \times_{X'} Y'$ in the normalization $f': Y' \rightarrow X'$ in V to obtain $Y'' \rightarrow Y'$. Further applying Lemma 2.4 and replacing X' by a resolution of the quotient Y''/G , we may assume that D' is regular.

We set $C \times_X X' = C' + F$ where C' is the normalization of C and F is a divisor supported on the inverse image E of u . We regard E as a reduced divisor of X' and let $M \geq 1$ denote an integer such that $(M-1)E \geq F$. For a reduced Cartier divisor C_1 satisfying $C' \equiv C'_1 \pmod{\mathfrak{m}_u^M}$, there exists a Cartier divisor C'_1 of X' such that $C_1 \times_X X' = C'_1 + F$. Since $C \times_X X' \equiv C_1 \times_X X' \pmod{\mathcal{I}_E^M}$, we have $C' \times_{X'} E = C'_1 \times_{X'} E$ and the divisor C'_1 is reduced and meets E transversely. Hence, it is a normalization of C_1 .

As we have shown above, there exists an integer $N' \geq 1$ such that the congruence $C' \equiv C'_2 \pmod{\mathfrak{m}_{u'}^{N'}}$ for a reduced Cartier divisor C'_2 of X' and for each point $u' \in C' \cap E$

imply the equality (2.2) holds. Set $N = M + N'$ and let C' be a closed reduced subscheme of X satisfying $C \equiv C' \bmod \mathfrak{m}_u^N$. Then, we have $C' \equiv C'_1 \bmod \mathcal{I}_E^{N'}$. Hence, we obtain the equality (2.2). \square

We show a continuity of the total dimension of the space of vanishing cycles.

Proposition 2.5. *Let C be a smooth curve over an algebraically closed field k of characteristic p and let $g: Y \rightarrow C$ be a smooth morphism of schemes over k of relative dimension 1. Let $f: X \rightarrow Y$ be a proper morphism of schemes over k . Let Λ be a finite field of characteristic $\ell \neq p$ and let \mathcal{F} be a constructible sheaf of Λ -modules on X locally acyclic relatively to $X \rightarrow C$. Let $Z \subset X$ be a closed subscheme such that the restriction of \mathcal{F} on $X - Z$ is universally locally acyclic relatively to $X \rightarrow Y$. Let B be a linear combination of divisors on Y flat over C supported on the closed subset $E = f(Z)$.*

Let u be a closed point of Z and set $v = f(u) \in Y$ and $s = g(v) \in C$. Assume that v is an isolated point of the intersection $E \cap Y_s$ and that u is the unique point in the intersection $f^{-1}(v) \cap Z_s$. Assume also that, for every closed point $t \in C, t \neq s$, the fiber $Z_t = Z \times_C t$ is finite and that, for every point $y \in E_t$, we have

$$(2.3) \quad \sum_{z \in Z_t, f(z)=y} \dim \text{tot } \phi_z(\mathcal{F}|_{X_t}, f|_{X_t}) = (B, Y_t)_y.$$

Then, the equality (2.3) holds also for $t = s$ and $y = v$.

Proof. By the assumption, E is quasi-finite over C on a neighborhood of v . Hence, by replacing C by an étale neighborhood of s and Y by an étale neighborhood of v , we may assume that E is finite over C and that v is the unique point of E above s . Then, Z is also finite over C and u is the unique point of Z above s . We define functions a and b on C by

$$a(t) = \sum_{z \in Z_t} \dim \text{tot}_z \phi(\mathcal{F}|_{X_t}, f|_{X_t}), \quad b(t) = (B, Y_t).$$

By the assumption, we have $a = b$ on $C - \{s\}$. By the assumption that B is flat over C , the function b is constant at s . Hence, it suffices to show that the function a is also constant at s .

Let $t \in C$ be a closed point. The complex of nearby cycles $\phi(\mathcal{F}|_{X_t}, f|_{X_t})$ is supported on Z_t by the assumption that the restriction of \mathcal{F} to $X - Z$ is universally locally acyclic relatively to $f: X \rightarrow Y$. For $y \in E_t$, let $\bar{\eta}_y$ be a geometric generic point of the strict localization of Y_t at y . Then, the distinguished triangle of vanishing cycles gives us a distinguished triangle

$$(2.4) \quad \rightarrow (Rf_* \mathcal{F}|_{Y_t})_y \rightarrow (Rf_* \mathcal{F}|_{Y_t})_{\bar{\eta}_y} \rightarrow \bigoplus_{z \in Z_t, f(z)=y} \phi_z(\mathcal{F}|_{X_t}, f|_{X_t}) \rightarrow .$$

Hence $a(t)$ equals the sum of the Artin conductors $\sum_{y \in E_t} a_y(Rf_* \mathcal{F}|_{Y_t})$ defined as (1.8).

We prove

$$(2.5) \quad a(s) - a(t) = -\dim \phi_v(Rf_* \mathcal{F}, g) = 0.$$

for $t \in C - \{s\}$ to complete the proof. The first equality is a consequence of the lemma below. We show the vanishing $\phi_v(Rf_* \mathcal{F}, g) = 0$. Since \mathcal{F} is assumed locally acyclic relatively to $X \rightarrow C$, the canonical morphism $\mathcal{F}_s \rightarrow \psi(\mathcal{F}, g \circ f)$ is an isomorphism on X_s .

Since the formation of the nearby cycle complex is compatible with proper push-forward, it implies that the canonical morphism $Rf_*\mathcal{F}_s \rightarrow \psi(Rf_*\mathcal{F}, g)$ is an isomorphism. Thus, we obtain the required vanishing $\phi_v(Rf_*\mathcal{F}, g) = 0$ and the equality (2.5) is proved. \square

Lemma 2.6. *Let $g: X \rightarrow C$ be a smooth morphism over an algebraically closed field k of characteristic p from a smooth surface X to a smooth curve C . Let D be a divisor of X finite flat over C and s be a closed point of C such that the closed fiber D_s consists of a unique point x . Let Λ be a finite field of characteristic $\ell \neq p$ and let \mathcal{K} be a constructible complex of Λ -modules on X such that the restriction $\mathcal{H}^q\mathcal{K}|_U$ of the cohomology sheaf on the complement $U = X - D$ is locally constant for every integer q .*

Then, on a neighborhood of s in C , the sum of the Artin conductors $\sum_{z \in D_t} a_z(\mathcal{K}|_{X_t})$ is constant except possibly for $s = t$ and satisfies

$$(2.6) \quad a_x(\mathcal{K}|_{X_s}) - \sum_{z \in D_t} a_z(\mathcal{K}|_{X_t}) = -\dim \phi_x(\mathcal{K}, g).$$

Proof. By devissage, it suffices to consider the case where $\mathcal{K} = j_!\mathcal{F}$ for a locally constant constructible sheaf \mathcal{F} on U and the open immersion $j: U \rightarrow X$ and the case where $\mathcal{K} = i_*\mathcal{G}$ for a constructible sheaf \mathcal{G} on D and the closed immersion $i: D \rightarrow X$. The first case is [19, Théorème 5.1.1]. The second case follows from the exact sequence $0 \rightarrow \mathcal{G}_x \rightarrow \bigoplus_{z \in D_{\bar{\eta}}} \mathcal{G}_z \rightarrow \phi_x(i_*\mathcal{G}, g) \rightarrow 0$ where $\bar{\eta}$ denotes a geometric generic point of the strict localization S of C at s . \square

2.2 An application of Elkik's theorem

To prove the stability of nearby cycle in the next subsection, we recall the following generalization of Hensel's lemma due to Elkik [10, Section 2], with slight reformulation. See also [25, 3.2.2]. Let $S = \operatorname{Spec} R$ be an affine noetherian scheme and $Y = \operatorname{Spec} B$ be an affine scheme of finite type over S . By taking a finite presentation $B = R[T]/(f)$ where T denotes a system of indeterminates and f denotes a system of polynomials, a closed subscheme Z of Y is defined by the ideal $H_B = \sum K_{(\alpha)}\Delta_{(\alpha)} \subset R[T]$ in the notation [10, 0.2]. As noted there, the support of Z is the largest open subscheme of Y smooth over S and the ideal can only get larger by base change. Although Z depends on presentation, let $Z_{Y/S}$ denote it by abuse of notation.

Lemma 2.7 ([10, Théorème 2]). *Let S be an affine noetherian scheme, $X = \operatorname{Spec} A$ be an affine noetherian scheme over S and let $J \subset A$ be an ideal such that the pair (A, J) is henselian. For an integer $n \geq 1$, set $X_n = \operatorname{Spec} A/J^n \subset X$. Let $h \geq 0$ be an integer.*

Then, there exist integers $m \geq r \geq 0, m \geq h$ such that, for any affine scheme Y over S of finite type and any morphism of schemes $\bar{f}: X_n \rightarrow Y$ over S for $n \geq m$ satisfying $Z_{Y/S} \times_Y X_n \subset X_h$, there exists a morphism $f: X \rightarrow Y$ over S that makes the diagram

$$(2.7) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup \uparrow & & \uparrow \bar{f} \\ X_{n-r} & \xrightarrow{\subset} & X_n \end{array}$$

commutative.

Proof. Since the ideal defining $Z_{Y/S}$ get larger by base change as remarked in [10, 0.2], we may assume $X = S$ by taking the base change by $X \rightarrow S$.

In the notation of [10, Théorème 2], the condition $J(\mathbf{a}^0) \subset \mathcal{J}^n$ means that a morphism $X_n \rightarrow Y$ is defined. Further, under this condition and $h \leq n$, the condition $H_B(\mathbf{a}^0) \supset \mathcal{J}^h$ means a closed immersion $Z \times_Y X_n \subset X_h$. Since the condition $J(\mathbf{a}^0) = 0$ means that a morphism $X \rightarrow Y$ is defined and since the congruence $\mathbf{a} \equiv \mathbf{a}^0 \pmod{\mathcal{J}^{n-r}}$ means the commutative diagram (2.7), the assertion follows by [10, Théorème 2]. \square

Proposition 2.8. *Let $f: X = \operatorname{Spec} A \rightarrow S$ be a morphism of finite type of affine noetherian schemes and X_1 be the closed subscheme defined by an ideal $I \subset A$. Assume that X is normal and that the complement $U = X - X_1$ is a dense open subscheme smooth over S . Let $\tilde{X} = \operatorname{Spec} \tilde{A}$ be the henselization of X along X_1 . Let $V \rightarrow U$ be a G -torsor for a finite group G and let Y be the normalization of X in V .*

Then, there exists integers $r \geq 0$ and $N \geq r + 2$ such that for a morphism $g: X \rightarrow S$ satisfying $g \equiv f \pmod{I^N}$, there exist isomorphisms $\tilde{p}: \tilde{X} \rightarrow \tilde{X}$ and $\tilde{q}: \tilde{Y} = Y \times_X \tilde{X} \rightarrow \tilde{Y}$ satisfying the following properties: The diagram

$$(2.8) \quad \begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{q}} & \tilde{Y} \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{p}} & \tilde{X} \\ & \searrow \tilde{g} \quad \swarrow \tilde{f} & \\ & S & \end{array}$$

where \tilde{f} and \tilde{g} denote the composition with f and g is commutative and compatible with the G -actions. They are congruent to the identity modulo $I^{N-r}\mathcal{O}_{\tilde{X}}$ and on $I^{N-r}\mathcal{O}_{\tilde{Y}}$ respectively.

Proof. For a scheme T over X , let T_f denote T regarded as a scheme over S with respect to the composition with $f: X \rightarrow S$ and similarly for T_g for a morphism $g: X \rightarrow S$. For an integer $n \geq 1$ and for a scheme T over X , let $T_n \subset T$ denote the closed subscheme $T \times_X \operatorname{Spec} A/I^n$. If $g: X \rightarrow S$ satisfies $g \equiv f \pmod{I^N}$ and if $n \leq N$, we have $T_{n,g} = T_{n,f}$ for a scheme T over X and we will drop the subscripts f and g in this case.

Let Z be the closed subscheme $Z_{X_f/S}$ of X . By the assumption that U is smooth over S , the intersection $Z \cap U$ is empty. Hence, there exists an integer $h \geq 1$ such that \mathcal{O}_Z is annihilated by I^{h-1} . Let $m \geq r \geq 0, m \geq h$ be integers as in Lemma 2.7 for the henselian pair $(\tilde{A}, I\tilde{A})$.

Let N be an integer satisfying $N \geq m$ and $N \geq 2 + r$. Let $g: X \rightarrow S$ be a morphism of schemes satisfying $g \equiv f \pmod{I^N}$. We apply Lemma 2.7 to the canonical immersion $\tilde{X}_N \rightarrow X_f$ over S . Since $N \geq h$, the assumption $Z \times_{X_f} X_N = Z \times_{X_f} X_{h-1} \subset X_h$ is satisfied by Nakayama's lemma. Hence we obtain a commutative diagram

$$(2.9) \quad \begin{array}{ccc} \tilde{X}_g & \xrightarrow{p} & X_f \\ \cup \uparrow & & \uparrow \\ \tilde{X}_{N-r} & \xrightarrow{\subset} & \tilde{X}_N \end{array}$$

of schemes over S . The induced morphism $\tilde{p}: \tilde{X}_g \rightarrow \tilde{X}_f$ on the henselizations induces the identity on $\tilde{X}_2 \subset \tilde{X}_{N-r}$ and hence is étale. Since \tilde{X} is henselian, $\tilde{p}: \tilde{X}_g \rightarrow \tilde{X}_f$ itself is an isomorphism.

Let Z' be the closed subscheme $Z_{Y/X}$ of Y . By the assumption that $V \rightarrow U$ is a G -torsor, the intersection $Z' \cap V$ is empty. Define an integer h' similarly as h above and let $m' \geq r' \geq 0, m' \geq h'$ be integers defined for $\tilde{Y}_f \rightarrow \tilde{X}_f$ in Lemma 2.7. Then, by a similar argument as above for $Y_f \rightarrow X_f$, there exists an integer $N' \geq N$ such that if $g \equiv f \pmod{I^{N'}}$, there exists an isomorphism $\tilde{q}: \tilde{Y}_g \rightarrow \tilde{Y}_f$ such that the diagram (2.8) is commutative.

We show that the morphism $q: \tilde{Y}_g \rightarrow \tilde{Y}_f$ is compatible with the action of G , after replacing N' by a larger integer if necessary. Let $n \geq 1$ be an integer such that the restriction map $\mathrm{Hom}_{\tilde{p}}(\tilde{Y}_g, \tilde{Y}_f) \rightarrow \mathrm{Hom}_{\tilde{p}}(\tilde{Y}_{g,n}, \tilde{Y}_f)$ is injective. Then, if $N \geq m, N \geq r + n$, the restrictions to $\tilde{Y}_n \subset \tilde{Y}_{N-r}$ of both compositions in the diagram

$$(2.10) \quad \begin{array}{ccc} \tilde{Y}_g & \xrightarrow{\tilde{q}} & \tilde{Y}_f \\ \sigma \downarrow & & \downarrow \sigma \\ \tilde{Y}_g & \xrightarrow{\tilde{q}} & \tilde{Y}_f \end{array}$$

for $\sigma \in G$ are the same. Hence the diagram (2.10) itself is commutative for $\sigma \in G$ and $\tilde{q}: \tilde{Y}_g \rightarrow \tilde{Y}_f$ is compatible with the action of G . \square

2.3 Stability of nearby cycles

The stability of nearby cycles at an isolated singularity is an immediate consequence of Proposition 2.8.

Proposition 2.9. *Let X be a scheme of finite type over a perfect field k of characteristic p , C be a smooth curve over k , and let $f: X \rightarrow C$ be a flat morphism over k . Let u be a closed point of X such that $U = X - \{u\}$ is smooth over C and $j: U \rightarrow X$ be the open immersion. Let Λ be a finite field of characteristic $\ell \neq p$ and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on U .*

Then, there exists an integer $N \geq 1$ such that, for a morphism $g: X \rightarrow C$ satisfying $g \equiv f \pmod{\mathfrak{m}_u^N}$, there exists an isomorphism

$$(2.11) \quad R\psi_u(j_!\mathcal{F}, f) \rightarrow R\psi_u(j_!\mathcal{F}, g).$$

Proof. Let \tilde{X} be the henselization of X at u and $\tilde{\mathcal{F}}$ be the pull-back of \mathcal{F} on $\tilde{U} = U \times_X \tilde{X}$. Then, by Proposition 2.8 applied to $X \rightarrow S$ and a finite Galois covering $V \rightarrow U$ trivializing \mathcal{F} , there exists an integer $N \geq 1$ such that, for a morphism $g: X \rightarrow C$ satisfying $g \equiv f \pmod{\mathfrak{m}_u^N}$, there exists an isomorphism $\tilde{p}: \tilde{X}_g \rightarrow \tilde{X}_f$ over C and an isomorphism $\tilde{p}^*: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$. They induce an isomorphism (2.11). \square

To prove the main result in this section, we show the vanishing of a certain limit of the space of vanishing cycles. We begin with the study of the limit of the local rings with respect to a sequence of blow-up.

Lemma 2.10 ([1, Proposition 1.9.4], [12, 5.4]). *Let A be a local ring, \mathfrak{p} be a prime ideal and let $f \in A$ be a non-zero divisor. Assume that $\bar{A} = A/\mathfrak{p}$ is a discrete valuation ring and that $\bar{f} \in \bar{A}$ is a uniformizer.*

1. *Let A' denote the subring $A[\mathfrak{p}/f^n; n \geq 1] \subset A[1/f]$. Then, $\mathfrak{p}' = \mathfrak{p}A[1/f]$ is a prime ideal of A' and the canonical morphism $A/\mathfrak{p} \rightarrow A'/\mathfrak{p}'$ is an isomorphism. We have $f\mathfrak{p}' = \mathfrak{p}'$ and the ideal fA' is a maximal ideal. The canonical morphism $A[1/f] \rightarrow A'[1/f]$ is an isomorphism.*

2. *Assume $f\mathfrak{p} = \mathfrak{p}$. Then the ring $A[1/f]$ equals the local ring $A_{\mathfrak{p}}$ and the canonical morphism $\mathfrak{p} \rightarrow \mathfrak{p}A_{\mathfrak{p}}$ is an isomorphism.*

3. *Assume that A is henselian and that $f\mathfrak{p} = \mathfrak{p}$. Then, the local ring $A_{\mathfrak{p}}$ is also henselian.*

We record a proof of 1. and 2. for the convenience of the reader.

Proof. 1. By the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{p} & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & 0 \\ & & \downarrow \cap & & \downarrow \cap & & \downarrow \cap & & \\ 0 & \longrightarrow & \mathfrak{p}A[1/f] & \longrightarrow & A[1/f] & \longrightarrow & \bar{A}[1/\bar{f}] & \longrightarrow & 0, \end{array}$$

the subring $A' = A + \mathfrak{p}A[1/f] \subset A[1/f]$ is the inverse image of $\bar{A} \subset \bar{A}[1/\bar{f}]$ by the surjection $A[1/f] \rightarrow \bar{A}[1/\bar{f}]$. Hence, we obtain an isomorphism $A'/\mathfrak{p}' \rightarrow \bar{A}$ and $\mathfrak{p}' = \mathfrak{p}A[1/f]$ is a prime ideal of A' . We have $f\mathfrak{p}' = f\mathfrak{p}A[1/f] = \mathfrak{p}A[1/f] = \mathfrak{p}'$ and $A'/fA' = (A'/\mathfrak{p}')/(f) = \bar{A}/\bar{f}\bar{A}$ is the residue field of A . The inclusions $A \rightarrow A' \rightarrow A[1/f]$ imply an isomorphism $A[1/f] \rightarrow A'[1/f]$.

2. We show that $A[1/f]$ is a local ring and that its maximal ideal is $\mathfrak{p}A[1/f]$. Let $g \in A$ and $n \geq 0$ be such that $g/f^n \in A[1/f]$ is not in $\mathfrak{p}A[1/f] = \text{Ker}(A[1/f] \rightarrow \bar{A}[1/\bar{f}])$. Since g is not contained in \mathfrak{p} and \bar{f} is a uniformizer of \bar{A} , it is of the form $g = uf^m + b$ for $u \in A^\times, m \geq 0, b \in \mathfrak{p}$. Writing $b = f^m c$ for $c \in \mathfrak{p}$, we obtain $g = f^m(u + c)$ and $u + c \in A$ is invertible. Hence $A[1/f]$ is a local ring and is equal to $A_{\mathfrak{p}}$.

Since $f\mathfrak{p} = \mathfrak{p}$ and $A[1/f] = A_{\mathfrak{p}}$, we have $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}A[1/f] = \mathfrak{p}$.

3. Let \tilde{B} be the local ring of an étale algebra over $A_{\mathfrak{p}}$ at a maximal ideal above \mathfrak{p} such that the residue field is isomorphic to the residue field $\kappa(\mathfrak{p})$ of the local ring $A_{\mathfrak{p}}$. We show that the canonical morphism $A_{\mathfrak{p}} \rightarrow \tilde{B}$ is an isomorphism.

By Zariski's main theorem, there exist a finite A -algebra B and an isomorphism $B_{\mathfrak{q}} \rightarrow \tilde{B}$ from the localization at a prime ideal \mathfrak{q} of B above \mathfrak{p} . By replacing B by the quotient by the \mathfrak{p} -torsion part, we may assume that the canonical morphism $B \rightarrow B \otimes_A A_{\mathfrak{p}}$ is an injection. The finite $\kappa(\mathfrak{p})$ -algebra $B \otimes_A \kappa(\mathfrak{p})$ is decomposed as $\kappa(\mathfrak{q}) \times C$. We identify the residue field $\kappa(\mathfrak{q})$ with $\kappa(\mathfrak{p})$ by the canonical isomorphism.

Let \bar{B}_1 be the image of B in C and set $\bar{B}' = A/\mathfrak{p} \times \bar{B}_1 \subset \kappa(\mathfrak{q}) \times C = B \otimes_A \kappa(\mathfrak{p})$. The kernel of the canonical surjection $B \otimes_A A_{\mathfrak{p}} \rightarrow B \otimes_A \kappa(\mathfrak{p})$ is the image of $B \otimes_A \mathfrak{p}A_{\mathfrak{p}}$ and is contained in B by 2. Since the cokernel of the canonical map $B \rightarrow \bar{B}'$ is an A -module of finite length, the inverse image B' of \bar{B}' by the surjection $B \otimes_A A_{\mathfrak{p}} \rightarrow B \otimes_A \kappa(\mathfrak{p})$ is a finite A -algebra and the canonical morphism $B \otimes_A A_{\mathfrak{p}} \rightarrow B' \otimes_A A_{\mathfrak{p}}$ is an isomorphism.

Since A is henselian, the finite A -algebra B' is the product of local rings. Thus, replacing B by the factor of B' whose spectrum contains \mathfrak{q} , we may assume that $\kappa(\mathfrak{p}) \rightarrow B \otimes_A \kappa(\mathfrak{p})$ is an isomorphism. Then, $A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is an isomorphism by Nakayama's lemma. \square

Proposition 2.11. *Let S be the spectrum of an excellent discrete valuation ring and X be a scheme of finite type over S . Let D be a closed regular integral subscheme of X finite and flat over S and let E be a Cartier divisor of X meeting D transversely. Let x be a closed point of $D \cap E$. For $n \geq 1$, let $X_n \rightarrow X$ denote the blow-up at $D \cap nE$, let x_n be the closed point above x of the proper transform of D and let \bar{x}_n be a geometric point of X_n above x_n .*

Let Λ be a finite field of characteristic ℓ invertible on S and \mathcal{F} be a constructible sheaf of Λ -modules on X . Then, for an integer $q > 0$, the inductive limit $\varinjlim_n R^q \psi_{\bar{x}_n} \mathcal{F}$ is zero.

Proof. Let \bar{S} and X_{n,\bar{x}_n} denote the strict localizations and $\bar{\eta}$ denote a geometric generic point of \bar{S} . Then, $\varprojlim_n X_{n,\bar{x}_n} \times_{\bar{S}} \bar{\eta}$ is strictly local by Lemma 2.10. Hence $\varinjlim_n R^q \psi_{\bar{x}_n} \mathcal{F} = H^q(\varprojlim_n X_{n,\bar{x}_n} \times_{\bar{S}} \bar{\eta}, \mathcal{F})$ is zero. \square

Lemma 2.12. *Let S be the spectrum of an excellent discrete valuation ring, X be a normal flat scheme of finite type over S of relative dimension 1. Let $D \subset X$ be a reduced closed subscheme of X finite and flat over S and let $j: U = X - D \rightarrow X$ denote the open immersion.*

Let $X' \rightarrow X$ be a proper birational morphism as in Lemma 2.4 such that the proper transform $D' \subset X'$ of D is regular and meets the reduced part E of the closed fiber X'_s transversely.

For $n \geq 1$, let $X_n \rightarrow X'$ be the composition with the blow-up at $nE \cap D'$ and let $D_n \subset X_n$ denote the proper transform of D' . Then for a geometric point \bar{x} of D_s and for a locally constant sheaf \mathcal{F} of Λ -modules on U_K , the canonical mapping

$$(2.12) \quad H_c^1((X_n - D_n) \times_X \bar{x}, R\psi_{j!} \mathcal{F}) \rightarrow R^1 \psi_{\bar{x}j!} \mathcal{F}$$

is injective. Further, there exists an integer $m \geq 1$ such that, for every $n \geq m$, the canonical mapping (2.12) is an isomorphism.

Proof. The canonical morphism $H^1(X_n \times_X \bar{x}, R\psi_{j!} \mathcal{F}) \rightarrow R^1 \psi_{\bar{x}j!} \mathcal{F}$ is an isomorphism by the proper base change theorem. Hence, the injectivity follows from the exact sequence (2.13)

$$\bigoplus_{\bar{x}' \in D_n \times_X \bar{x}} R^0 \psi_{\bar{x}'j!} \mathcal{F} \rightarrow H_c^1((X_n - D_n) \times_X \bar{x}, R\psi_{j!} \mathcal{F}) \rightarrow R^1 \psi_{\bar{x}j!} \mathcal{F} \rightarrow \bigoplus_{\bar{x}' \in D_n \times_X \bar{x}} R^1 \psi_{\bar{x}'j!} \mathcal{F}$$

and $R^0 \psi_{\bar{x}'j!} \mathcal{F} = 0$ for $\bar{x}' \in D_n \times_X \bar{x}$. By Proposition 2.11, the inductive limit \varinjlim_n of the last term in (2.13) is zero. Since $R^1 \psi_{\bar{x}j!} \mathcal{F}$ is of finite dimension, there exists an integer $m \geq 1$ such that the last map in (2.13) is the zero-map for $n \geq m$. Hence, for $n \geq m$, the second arrow in the exact sequence (2.13) is an isomorphism. \square

We prove the following stability of nearby cycles for a fibration from a surface to a curve. A similar stability is proved by Laumon in [21, Théorème 6.1.4] in arbitrary dimension, under the assumption that the normalization of a covering trivializing the sheaf has an isolated singularity.

Theorem 2.13. *Let X be a normal surface and C be a smooth curve over a perfect field k of characteristic p , and let $f: X \rightarrow C$ be a flat morphism over k . Let D be a closed subscheme of X and $j: U = X - D \rightarrow X$ be the open immersion. Let Λ be a finite field of characteristic $\ell \neq p$ and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on U .*

Let u be a closed point of X and such that u is an isolated characteristic point of $f: X \rightarrow C$ with respect to $j_! \mathcal{F}$ and that $D - \{u\}$ is étale over C .

1. There exists an integer $N \geq 1$ such that, for a morphism $g: X \rightarrow C$ satisfying $g \equiv f \pmod{\mathfrak{m}_u^N}$, we have an equality

$$(2.14) \quad \dim R^1 \psi_u(j_! \mathcal{F}, f) = \dim R^1 \psi_u(j_! \mathcal{F}, g).$$

2. There exists an integer $N \geq 1$ such that, for a morphism $g: X \rightarrow C$ satisfying $g \equiv f \pmod{\mathfrak{m}_u^N}$, there exists an isomorphism

$$(2.15) \quad R^1 \psi_u(j_! \mathcal{F}, f) \rightarrow R^1 \psi_u(j_! \mathcal{F}, g).$$

Proof. By Proposition 2.9, it suffices to prove the case where u is in the closure of $D - \{u\}$. By shrinking X , we may assume D is flat over C and u is the unique point of the fiber of $D \rightarrow C$.

1. First, we deduce the case where $\mathcal{F} = \Lambda_U$ from Proposition 2.9. Let $i: D \rightarrow X$ denote the closed immersion. By the exact sequence $0 \rightarrow j_! \Lambda_U \rightarrow \Lambda_X \rightarrow i_* \Lambda_D \rightarrow 0$ and $R^q \psi_u(j_! \Lambda_U, f) = 0$ for $q \neq 1$, we have an equality

$$(2.16) \quad \dim R^1 \psi_u(j_! \Lambda_U, f) = \dim R\psi_u(\Lambda_X, f) - \dim R\psi_u(\Lambda_D, f|_D).$$

By the assumption, $X - \{u\} \rightarrow C$ is smooth and $D - \{u\} \rightarrow C$ is étale. Hence, by Proposition 2.9, there exists an integer $N \geq 1$ such that, for a morphism $g: X \rightarrow C$ satisfying $g \equiv f \pmod{\mathfrak{m}_u^N}$, we have isomorphisms $R\psi_u(\Lambda_X, f) \rightarrow R\psi_u(\Lambda_X, g)$ and $R\psi_u(\Lambda_D, f|_D) \rightarrow R\psi_u(\Lambda_D, g|_D)$. Further, we have an equality (2.16) with f replaced by g by Proposition 2.1. Hence the equality (2.14) holds for $\mathcal{F} = \Lambda_U$.

We prove the general case. Let $\mathcal{F}_0 = \mathcal{F} - \text{rank } \mathcal{F} \cdot \Lambda_U$ denote the virtual difference. Let S denote the strict localization of C at a geometric point \bar{s} above $f(u)$ and let $\bar{\eta}$ be a geometric point of S defined by an algebraic closure of the fraction field. Then, we have

$$\begin{aligned} & \dim R^1 \psi_u(j_! \mathcal{F}, f) - \text{rank } \mathcal{F} \cdot \dim R^1 \psi_u(j_! \Lambda_U, f) \\ &= \sum_{x \in (D, f) \times_S \bar{\eta}} \dim \text{tot}_x(j_! \mathcal{F}_0|_{(U, f) \times_S \bar{\eta}}) - \dim \text{tot}_u(j_! \mathcal{F}_0|_{(U, f) \times_S \bar{s}}) \end{aligned}$$

and similarly for $\dim R^1 \psi_u(j_! \mathcal{F}, g)$ by [19, Théorème 5.1.1], [17, Theorem (6.7)], [14, Theorem 11.9]. By Propositions 2.1 and 2.3 and by what we have proved above, there exists an integer $N \geq 1$ such that $g \equiv f \pmod{\mathfrak{m}_u^N}$ implies

$$(2.17) \quad \sum_{x \in (D, f) \times_S \bar{\eta}} \dim \text{tot}_x(j_! \mathcal{F}|_{(U, f) \times_S \bar{\eta}}) = \sum_{x \in (D, g) \times_S \bar{\eta}} \dim \text{tot}_x(j_! \mathcal{F}|_{(U, g) \times_S \bar{\eta}}),$$

$$(2.18) \quad \dim \text{tot}_u(j_! \mathcal{F}|_{(U, f) \times_S \bar{s}}) = \dim \text{tot}_u(j_! \mathcal{F}|_{(U, g) \times_S \bar{s}})$$

respectively and the equality (2.14) for $\mathcal{F} = \Lambda_U$. They imply the equality (2.14).

2. By Lemma 2.12, there exists an integer $n \geq 0$ such that the morphism

$$(2.19) \quad H_c^1((X_n - D_n) \times_X \bar{u}, R\psi(j_! \mathcal{F}, f)) \rightarrow R^1 \psi_u(j_! \mathcal{F}, f)$$

is an isomorphism in the notation loc. cit. Changing the notation, let X' and D' denote X_n and D_n . Further by Lemma 2.12, the canonical morphism

$$(2.20) \quad H_c^1((X' - D') \times_X \bar{u}, R\psi(j_! \mathcal{F}, g)) \rightarrow R^1 \psi_u(j_! \mathcal{F}, g)$$

is an injection. Thus, by 1. it suffices to show that there exists an integer $N \geq 1$ such that for a morphism $g: X \rightarrow C$ satisfying $g \equiv f \pmod{\mathfrak{m}_u^N}$, there exists an isomorphism

$$(2.21) \quad H_c^1((X' - D') \times_X \bar{u}, R\psi(j_! \mathcal{F}, f)) \rightarrow H_c^1((X' - D') \times_X \bar{u}, R\psi(j_! \mathcal{F}, g)).$$

To apply Proposition 2.8, we construct a contraction $X' \rightarrow X'' \rightarrow X$ as follows. By shrinking C and X , we may assume that C and X are affine. Let $\pi: X' \rightarrow X$ be the canonical morphism. There exists an integer $m \geq 1$ such that $\pi^* \pi_* \mathcal{O}_{X'}(mD') \rightarrow \mathcal{O}_{X'}(mD')$ is surjective by an argument similar to that of Raynaud in the proof of [11, Lemme A]. Let X'' be the Stein factorization of $X' \rightarrow \text{Proj}_X \bigoplus_{m \geq 0} \pi_* \mathcal{O}_{X'}(mD')$ and let $\varphi: X'' \rightarrow X$ be the canonical morphism. Then, further as in the proof of loc. cit., the restriction $X'' - \varphi^{-1}(u) \rightarrow X - \{u\}$ of the proper morphism $\varphi: X'' \rightarrow X$ is an isomorphism and the morphism $X' \rightarrow X''$ contracts exactly those components of $\pi^{-1}(u)$ not meeting D' .

Since $X' \rightarrow X''$ is an isomorphism on a neighborhood of D' , we identify D' as a divisor of X'' . Then, by the proper base change theorem, the canonical morphism $H_c^1((X'' - D') \times_X \bar{u}, R\psi(j_! \mathcal{F}, f)) \rightarrow H_c^1((X' - D') \times_X \bar{u}, R\psi(j_! \mathcal{F}, f))$ is an isomorphism and the same for g . Thus to define an isomorphism (2.21), we may replace X' by X'' .

By the assumption that $f: X - \{u\} \rightarrow C$ is smooth, the restriction of f to U is smooth. Hence, the restriction of f to the complement $(X'' - D') - \varphi^{-1}(u)$ is also smooth. The divisor D' is φ -ample and the complement $X'' - D'$ is a scheme affine over X and hence is an affine scheme. Let $V \rightarrow U$ be a G -torsor for a finite group G such that the pull-back of \mathcal{F} on V is constant.

We apply Proposition 2.8 to the composition $X'' - D' \rightarrow X \rightarrow C$ and to the pull-back of the G -torsor $V \rightarrow U$. Let \tilde{X}'' be the henselization of $X'' - D'$ along the inverse image $\varphi^{-1}(u)$ and let \tilde{X}_f'' and \tilde{X}_g'' denote the scheme \tilde{X}'' regarded as schemes over C with respect to the compositions of $\tilde{X}'' \rightarrow X$ with $f: X \rightarrow C$ and $g: X \rightarrow C$ respectively, as in the proof of Proposition 2.8. Then, we obtain an isomorphism $h: \tilde{X}_g'' \rightarrow \tilde{X}_f''$ together with an isomorphism $h^*(j_! \mathcal{F}) \rightarrow j_! \mathcal{F}$ on \tilde{X}_g'' . They induce an isomorphism (2.21) with X' replaced by X'' as required. \square

3 Radon transform and the characteristic cycle

3.1 Preliminaries on the universal family of hyperplane sections

For the formalism of dual variety, we refer to [18]. Let X be a normal projective irreducible scheme over an algebraically closed field k of characteristic $p > 0$ and let \mathcal{L} be a very ample invertible \mathcal{O}_X -module. Let

$$X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee) = \text{Proj}_k S^\bullet E$$

be the closed immersion defined by \mathcal{L} to the projective space associated to the dual E^\vee of the k -vector space $E = \Gamma(X, \mathcal{L})$. We use an anti-Grothendieck notation to denote a projective space $\mathbf{P}(E)(k) = (E - \{0\})/k^\times$.

Let $\mathbf{P}^\vee = \mathbf{P}(E)$ be the dual of \mathbf{P} . The universal hyperplane $\mathbf{H} = \{(x, H) \mid x \in H\} \subset \mathbf{P} \times \mathbf{P}^\vee$ is defined by the identity $\text{id} \in \text{End}(E)$ regarded as a section $F \in \Gamma(\mathbf{P} \times \mathbf{P}^\vee, \mathcal{O}(1, 1)) = E \otimes E^\vee$. By the canonical injection $\Omega_{\mathbf{P}/k}^1(1) \rightarrow E \otimes \mathcal{O}_{\mathbf{P}}$, the universal hyperplane \mathbf{H} is identified with the covariant projective space bundle $\mathbf{P}(T^* \mathbf{P})$ associated

to the cotangent bundle $T^*\mathbf{P}$. Further, the identity of \mathbf{H} is the same as the map $\mathbf{H} = \mathbf{P}(T_{\mathbf{H}}^*(\mathbf{P} \times \mathbf{P}^\vee)) \rightarrow \mathbf{H} = \mathbf{P}(T^*\mathbf{P})$ induced by the locally splitting injection $\mathcal{N}_{\mathbf{H}/\mathbf{P} \times \mathbf{P}^\vee} \rightarrow \mathrm{pr}_1^* \Omega_{\mathbf{P}/k}^1$.

The fibered product $X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ is the intersection of $X \times \mathbf{P}^\vee$ with \mathbf{H} in $\mathbf{P} \times \mathbf{P}^\vee$ and is the universal family of hyperplane sections. We consider the universal family of hyperplane sections $p: X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$.

Assume that X is smooth of dimension d and $X \subsetneq \mathbf{P}$. We say that a reduced closed subscheme $T \subset T^*X$ is a linear subscheme if it is stable under the addition and the multiplication by scalars. For a reduced closed linear subscheme $T \subset T^*X$, we define a reduced subscheme

$$(3.1) \quad P(T) \subset X \times_{\mathbf{P}} \mathbf{H}$$

as follows. First, we consider the inverse image by the canonical surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \rightarrow T^*X$ and its restriction to the complement $X \times_{\mathbf{P}} (T^*\mathbf{P} - T_{\mathbf{P}}^*\mathbf{P}) \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ of the 0-section. Then, $P(T)$ is defined to be the unique reduced closed subscheme of $X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ such that its pull-back by the canonical projection $X \times_{\mathbf{P}} (T^*\mathbf{P} - T_{\mathbf{P}}^*\mathbf{P}) \rightarrow \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ is equal to the restriction to the complement of the 0-section.

Assume that a reduced closed linear subscheme T of T^*X is of codimension $d = \dim X$. Then, since $P(T) \subset X \times_{\mathbf{P}} \mathbf{H}$ is also of codimension d and $p: X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ is of relative dimension $d - 1$, the image $p(P(T)) \subset \mathbf{P}^\vee$ is of codimension at least 1.

Lemma 3.1. *Let X be a projective smooth scheme of dimension d and let \mathcal{L} be an ample invertible \mathcal{O}_X -module.*

1. *Assume that \mathcal{L} is very ample and satisfies the following condition:*

(L) *For every pair of distinct closed points $u \neq v$ of X , the canonical mapping*

$$(3.2) \quad E = \Gamma(X, \mathcal{L}) \rightarrow \mathcal{L}_u/\mathfrak{m}_u^2 \mathcal{L}_u \oplus \mathcal{L}_v/\mathfrak{m}_v^2 \mathcal{L}_v$$

is a surjection.

*Then, for an irreducible closed linear subscheme $T \subset T^*X$ of codimension $d = \dim X$, either the morphism $P(T) \rightarrow p(P(T))$ induced by $p: X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ is generically radicial or $p(P(T)) \subset \mathbf{P}^\vee$ is of codimension ≥ 2 . For another irreducible closed linear subscheme $T' \subset T^*X$ of codimension d , the intersection $p(P(T)) \cap p(P(T')) \subset \mathbf{P}^\vee$ is of codimension ≥ 2 if T and T' have no common irreducible components.*

2. *There exists an integer m such that $\mathcal{L}^{\otimes n}$ is very ample and satisfies the condition (L) for every $n \geq m$.*

Proof. 1. Let $\mathcal{I}_\Delta \subset \mathcal{O}_{X \times X}$ denote the ideal sheaf defining the diagonal immersion $\Delta: X \rightarrow X \times X$. Let $Z \subset X \times X$ be the closed subscheme defined by \mathcal{I}_Δ^2 and $p_1, p_2: Z \rightarrow X$ be the restriction of the projections. Define a vector bundle V over X and a line bundle L associated to a locally free \mathcal{O}_X -module $\tilde{\mathcal{L}} = p_{1*}p_2^* \mathcal{L}$ of rank $d + 1$ and the invertible \mathcal{O}_X -module \mathcal{L} respectively. The canonical isomorphism $\mathcal{L} \otimes \Omega_X^1 \rightarrow \mathcal{L} \otimes (\mathcal{I}_\Delta/\mathcal{I}_\Delta^2) \subset \tilde{\mathcal{L}}$ induces an injection

$$(3.3) \quad T^*X \otimes L \rightarrow V$$

of vector bundles. The cokernel of (3.3) is the line bundle L .

The condition (L) means that on $(X \times X)^\circ = X \times X - \Delta_X$, the canonical morphism $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_{(X \times X)^\circ} \rightarrow \text{pr}_1^* \tilde{\mathcal{L}} \oplus \text{pr}_2^* \tilde{\mathcal{L}}$ is a surjection and defines a surjection

$$(3.4) \quad E \times (X \times X)^\circ \rightarrow V \times_X (X \times X)^\circ \times_X V$$

of vector bundles on $(X \times X)^\circ$.

The images of twists by L of $T, T' \subset T^*X$ by (3.3) define closed subschemes $T \otimes L$ and $T' \otimes L$ of V . Then, the pull-backs of $T \otimes L$ and $T' \otimes L$ define a closed subscheme of $V \times_X (X \times X)^\circ \times_X V$. Pulling-back by (3.4) and applying the construction similar to the definition of $P(T)$ in (3.1), we define a closed subscheme $R(T, T')$ of $\mathbf{P}^\vee \times (X \times X)^\circ$. It consists of triples (H, u, v) of a hyperplane containing points $u \neq v$ such that $(u, H), (v, H) \in \mathbf{H}$ are contained in $P(T)$ and $P(T')$ respectively. Since $T, T' \subset T^*X$ are of codimension d , the codimension of $R(T, T') \subset \mathbf{P}^\vee \times (X \times X)^\circ$ is $2(d+1)$. Hence, the codimension of its image $S(T, T') \subset \mathbf{P}^\vee$ by the projection is at least $2(d+1) - 2d = 2$.

Assume $T = T'$ and let $H \in \mathbf{P}^\vee$ be a hyperplane not contained in $S(T, T)$. Then, there exists no two distinct points $u \neq v$ in X such that both (u, H) and (v, H) are contained in $P(T)$. In other words, $p(P(T)) \rightarrow \mathbf{P}^\vee$ is radicial outside $S(T, T)$.

Assume $T \neq T'$ and let $H \in \mathbf{P}^\vee$ be a hyperplane not contained in $S(T, T')$. Then, there exists no two distinct points $u \neq v$ such that (u, H) is in $P(T)$ and (v, H) is in $P(T')$. In other words, the intersection $p(P(T)) \cap p(P(T')) \subset \mathbf{P}^\vee$ is contained in the union of $S(T, T')$ and the image $p(P(T) \cap P(T'))$. By the assumption that T and T' have no common irreducible components, the intersection $P(T) \cap P(T') \subset X \times_{\mathbf{P}} \mathbf{H}$ is of codimension $\geq d+1$ and its image $p(P(T) \cap P(T')) \subset \mathbf{P}^\vee$ is of codimension $\geq (d+1) - (d-1) = 2$. Hence the assertion follows.

2. Define $Z \subset X \times X$ as in the proof of 1. Then, it suffices to apply the following Lemma to the complement $S = X \times X - \Delta_X$ of the diagonal and to the closed subscheme $\text{pr}_{12}^* Z \amalg \text{pr}_{13}^* Z$ of a proper flat scheme $X \times S \subset X \times X \times X$ over S . \square

Lemma 3.2. *Let S be a noetherian scheme, $f: X \rightarrow S$ be a proper flat scheme over S and \mathcal{L} be an f -ample invertible \mathcal{O}_X -module. For a closed subscheme Z of X flat over S , there exists an integer m such that for every $n \geq m$ and for every point $s \in S$, the restriction*

$$(3.5) \quad \Gamma(X_s, \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{X_s}) \rightarrow \Gamma(Z_s, \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{Z_s})$$

is a surjection.

Proof. Let $\mathcal{I}_Z \subset \mathcal{O}_X$ be the ideal sheaf defining Z . Since \mathcal{L} is f -ample, there exists an integer m such that for every $n \geq m$ and for every $q > 0$, we have $R^q f_* \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n} = 0$. For $n \geq m$, we have $H^1(X_s, \mathcal{I}_{Z_s} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{X_s}) = 0$ and (3.5) is a surjection. \square

The following lemma will be used to show the existence of a pencil defining a fibration close to f .

Lemma 3.3. *Let X be a projective normal scheme of dimension d and \mathcal{L} be an ample invertible \mathcal{O}_X -module. Let u be a closed point of X such that $X^\circ = X - \{u\}$ is smooth and let $N \geq 1$ be an integer.*

1. Assume that \mathcal{L} is very ample and satisfies the condition:

(N) For every point $x \in X, x \neq u$, the canonical morphism

$$(3.6) \quad E = \Gamma(X, \mathcal{L}) \rightarrow \mathcal{L}_u / \mathfrak{m}_u^N \mathcal{L}_u \oplus \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x$$

is a surjection.

Let $l_\infty \in E = \Gamma(X, \mathcal{L})$ be a non-zero section such that the hyperplane section X_∞ defined by l_∞ does not contain u . Let T_1, \dots, T_m be a finite family of closed linear irreducible subschemes of T^*X° of codimension d . Then, for $f \in \mathcal{O}_{X,u}/\mathfrak{m}_u^N$, there exists a non-zero section $l \in E = \Gamma(X, \mathcal{L})$ such that $l \neq l_\infty$,

$$(3.7) \quad l/l_\infty \equiv f \pmod{\mathfrak{m}_u^N},$$

that the hyperplane section X_0 of X defined by $l = 0$ is smooth outside u and that the intersection $T_{X_0}^*X^\circ \cap T_i$ with the conormal bundle of $X_0^\circ = X_0 - \{u\} \subset X^\circ$ is contained in the 0-section for every $i = 1, \dots, m$.

2. There exists an integer $m \geq 0$ such that $\mathcal{L}^{\otimes n}$ is very ample and satisfies the condition (N) for every $n \geq m$.

Proof. 1. We regard the k -vector space $W = \mathcal{O}_{X,u}/\mathfrak{m}_u^N$ as an affine space over k and let $E_f \subset E = \Gamma(X, \mathcal{L})$ denote the inverse image of $f \pmod{\mathfrak{m}_u^N}$ by the surjection $E = \Gamma(X, \mathcal{L}) \rightarrow W = \mathcal{O}_{X,u}/\mathfrak{m}_u^N$ sending l to $l/l_\infty \pmod{\mathfrak{m}_u^N}$.

We define a closed subscheme $Z \subset X^\circ \times X^\circ$, a vector bundle V of rank $d+1$ over X° and an injection $T^*X^\circ \otimes L \rightarrow V$ (3.3) of vector bundles of codimension 1 on X° similarly as in the proof of Lemma 3.1.1. We consider the pull-back $E \otimes \mathcal{O}_Z \rightarrow p_2^*\mathcal{L}$ of the canonical morphism $E \otimes \mathcal{O}_{X^\circ} \rightarrow \mathcal{L}$. Since $E \otimes \mathcal{O}_Z = p_1^*(E \otimes \mathcal{O}_{X^\circ})$, it induces $E \otimes \mathcal{O}_{X^\circ} \rightarrow p_{1*}p_2^*\mathcal{L}$ by adjunction and hence $E \times X^\circ \rightarrow V$. We define a surjection

$$(3.8) \quad E \times X^\circ \rightarrow V \times W$$

of vector bundles on X° to be its product with the canonical morphism $E \rightarrow W$.

We put $T_0 = T_{X_0}^*X^\circ$ and for each T_i , let $T_i \otimes L \subset V$ denote the image of the twist of T_i by L by (3.3) and define

$$E_{f,i} \subset E_f \times X^\circ$$

to be the inverse image of $(T_i \otimes L) \times \{f\}$ by (3.8). For $(l, x) \in E_f \times X^\circ$ such that $l \neq 0$, the condition $(l, x) \in E_{f,0}$ is equivalent to that x is a singular point of the hyperplane section X_0° defined by $l = 0$. Further, for $(l, x) \in E_f \times X^\circ$ not contained in $E_{f,0}$ and for $i \neq 0$, the condition $(l, x) \in E_{f,i}$ is equivalent to that the fiber of the line bundle $T_{X_0}^*X^\circ$ at x is not contained in T_i .

Consequently, $l \in E_f, \neq 0$ is not in the image of $E_{f,0}$ by the projection $E_f \times X^\circ \rightarrow E_f$ if and only if X_0° is smooth. Further, for such l , it is not in the image of $E_{f,i}$ if and only if the intersection $T_{X_0}^*X^\circ \cap T_i$ is contained in the 0-section. Thus, the hyperplane section X_0 satisfies the condition if and only if $l \in E_f, \neq 0$ is not in the union of the images of $E_{f,0}, \dots, E_{f,m}$ by the projection $E_f \times X^\circ \rightarrow E_f$.

The linear subscheme $T_0 = T_{X_0}^*X^\circ \subset T^*X^\circ$ is of codimension $d = \dim X$ and $T_i \subset T^*X^\circ$ for $i = 1, \dots, m$ are assumed to be of codimension d . Since the morphism (3.8) is surjective and the injection (3.3) is of codimension 1, the subvariety $E_{f,i} \subset E_f \times X^\circ$ is of codimension $d+1$. The images of $E_{f,i}$ by the projection $E_f \times X^\circ \rightarrow E_f$ are of codimension at least 1 in E_f and the assertion is proved.

2. Let $P \subset X \times X$ be the closed subscheme defined by \mathcal{I}_X^2 and let $T \subset X$ be the closed subscheme defined by \mathfrak{m}_u^N . Then, it suffices to apply Lemma 3.2 to $S = X - \{u\}$ and the closed subscheme $Z = (T \times S) \amalg (P \cap (X \times S))$ of $X \times X$. \square

Combining Lemmas 3.1 and 3.3, we obtain the following.

Proposition 3.4. *Let X be a projective irreducible smooth scheme of dimension d over an algebraically closed field k and \mathcal{L} be an ample invertible \mathcal{O}_X -module. Let T be a linear irreducible closed subscheme of T^*X of codimension d . Then, there exists an integer $m \geq 0$ such that for every $n \geq m$, the invertible \mathcal{O}_X -module $\mathcal{L}^{\otimes n}$ is very ample and satisfies the condition (L) in Lemma 3.1 and the morphism $P(T) \rightarrow p(P(T))$ is generically radicial.*

Proof. By Lemma 3.1, it suffices to show the existence of an integer m such that for $n \geq m$, there exists a hyperplane $H_0 \in \mathbf{P}^\vee$ such that the intersection of the fiber $(X \cap H_0) \times \{H_0\} = p^{-1}(H_0)$ with $P(T)$ for $\mathcal{L}^{\otimes n}$ consists of a unique point.

Let u be a closed point of X in the image of T by the canonical map $T^*X \rightarrow X$. Since T is of codimension d , there exists a function f defined on a neighborhood of u such that T and the section df of T^*X meet at a closed point of T^*X above u .

By Lemma 3.3.2, there exists an integer m such that for $n \geq m$, the invertible \mathcal{O}_X -module $\mathcal{L}^{\otimes n}$ satisfies the condition (N) in Lemma 3.3 for u and $N = 2$. Then by Lemma 3.3.1, for an integer $n \geq m$, there exist non-zero sections $l_\infty, l \in \Gamma(X, \mathcal{L}^{\otimes n})$, $l_\infty \neq l$ such that the hyperplane section X_∞ defined by l_∞ does not contain u and $l/l_\infty \equiv f \pmod{\mathfrak{m}_u^2}$, that the hyperplane section X_0 of X defined by $l = 0$ is smooth outside u and that the intersection $T_{X_0}^* X^\circ \cap T$ with the conormal bundle of $X_0^\circ = X_0 - \{u\} \subset X^\circ = X - \{u\}$ is contained in the 0-section.

Let H_0 be the hyperplane defined by $l = 0$ and g be the function l/l_∞ defined on $X - X_\infty$. Then the congruence $l/l_\infty \equiv f \pmod{\mathfrak{m}_u^2}$ implies that $dg(u) = df(u)$ in T_u^*X . Hence, the pair $(u, H_0) \in X \times_{\mathbf{P}} \mathbf{H}$ is a point of $P(T)$. Further, the conditions that X_0° is smooth and that $T_{X_0}^* X^\circ \cap T$ is contained in the 0-section imply that the intersection of the fiber $X_0 \times \{H_0\}$ of $p: X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ at H_0 with $P(T)$ is a subset of $\{u\}$. Thus, u is the unique point of the fiber $P(T) \rightarrow p(P(T))$. \square

Let $\mathbf{G} = \text{Gr}(1, \mathbf{P}^\vee)$ be the Grassmannian variety parametrizing lines in \mathbf{P}^\vee . The universal line $\mathbf{D} \subset \mathbf{G} \times \mathbf{P}^\vee$ is canonically identified with the flag variety parametrizing pairs (H, L) of points H of \mathbf{P}^\vee and lines L passing through H . We also identify \mathbf{D} with the projective space bundle $\mathbf{P}(T\mathbf{P}^\vee)$ associated to the tangent bundle of \mathbf{P}^\vee . We define $X_{\mathbf{G}}$ by the cartesian diagram

$$(3.9) \quad \begin{array}{ccc} X_{\mathbf{G}} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} \\ \downarrow & & \downarrow p \\ \mathbf{D} & \longrightarrow & \mathbf{P}^\vee \\ \downarrow & & \\ \mathbf{G} & & \end{array}$$

Let $\mathbf{A} \subset \mathbf{P} \times \mathbf{G} \rightarrow \mathbf{G}$ denote the universal axis. Then, $X \times_{\mathbf{P}} \mathbf{A}$ is the intersection $\mathbf{A} \cap (X \times \mathbf{G})$ and is proper smooth over X . The immersion $X \times_{\mathbf{P}} \mathbf{A} \rightarrow X \times \mathbf{G}$ is a regular immersion of codimension 2 and $X_{\mathbf{G}} \rightarrow X \times \mathbf{G}$ is the blow-up at $X \times_{\mathbf{P}} \mathbf{A}$.

For a line $L \subset \mathbf{P}^\vee$, we define X_L by the cartesian diagram

$$(3.10) \quad \begin{array}{ccc} X_L & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} \\ p_L \downarrow & & \downarrow p \\ L & \longrightarrow & \mathbf{P}^\vee. \end{array}$$

It is equal to $\{(x, H) \mid x \in X, H \in L, x \in H\}$. If the axis $A_L = \bigcap_{H \in L} H$ of L meets X transversely, then X_L is the blow up of X at the intersection $X \cap A_L$.

Let $T \subset T^*X$ be a linear reduced closed subscheme of codimension d and u be a closed point of X . Let f be a morphism to a smooth curve C over k defined on a neighborhood of u and assume that the intersection of T with the image of $df: X \times_C T^*C \rightarrow T^*X$ is contained in the fiber of u on a neighborhood of u . Then, for a basis ω of $X \times_C T^*C$ on a neighborhood of u , the intersection number $(T, [\omega])_{T^*X, u}$ with the image of the section of T^*X defined on a neighborhood of u is defined and is independent of the choice of ω . More intrinsically, it is the intersection product of the twist $\text{Hom}(X \times_C T^*C, T)$ with the image of the section df of the twisted vector bundle $\text{Hom}(X \times_C T^*C, T^*X)$ at the inverse image of u and we will write it as

$$(3.11) \quad (T, [df])_{T^*X, u}$$

by abuse of notation.

Lemma 3.5. *Let $T \subset T^*X$ be a linear closed subscheme of dimension $d = \dim X$ and L be a line in \mathbf{P}^\vee . Assume that the axis A_L meets X transversely and that T is contained in the 0-section T_X^*X on a neighborhood of $X \cap A_L$.*

*Let u be a closed point of X not in $X \cap A_L$ and set $v = p_L(u)$. Assume that, on a neighborhood of $T^*X \times_X (p_L^{-1}(v) - (X \cap A_L))$, the intersection of T with the image of $T^*L \rightarrow T^*X$ is contained in the fiber T_u^*X of u .*

Then, v is an isolated point of the intersection $p_(P(T)) \cap L \subset \mathbf{P}^\vee$ if v is contained in it and we have*

$$(3.12) \quad (p_*(P(T)), L)_{\mathbf{P}^\vee, v} = (T, [dp_L])_{T^*X, u}.$$

Proof. By the projection formula and by the assumption on the intersection, we have $(p_*(P(T)), L)_{\mathbf{P}^\vee, v} = (P(T), X_L)_{X \times_{\mathbf{P}} \mathbf{H}, u}$. On the complement of $X \cap A_L$, the kernel of the surjection $\Omega_{X/k}^1 \rightarrow \Omega_{X/L}^1$ is the image of $p_L^* \Omega_{L/k}^1$. Hence the immersion $X - (X \cap A_L) \rightarrow \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ induced by the restriction of p_L is defined by the canonical injection $T^*L \times_L (X - X \cap A_L) \rightarrow T^*\mathbf{P} \times_{\mathbf{P}} (X - X \cap A_L)$. Let \tilde{T} be the inverse image of T by the surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \rightarrow T^*X$ appeared in the definition (3.1) of $P(T)$. For a basis ω of $X \times_L T^*L$ on a neighborhood of u , we have $(P(T), X_L)_{X \times_{\mathbf{P}} \mathbf{H}, u} = (\tilde{T}, [\omega])_{X \times_{\mathbf{P}} T^*\mathbf{P}, u}$ by the definition of $P(T)$. Further, the right hand side is equal to $(T, [\omega])_{T^*X, u}$. \square

3.2 Radon transform and vanishing cycles

Let X be a smooth projective connected surface over an algebraically closed field k of characteristic $p > 0$. Let $D \subsetneq X$ be a reduced closed subscheme of X and $j: U = X - D \rightarrow X$ be the open immersion of the complement. Let Λ be a finite field of characteristic $\ell \neq p$ and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on $U = X - D$.

Let \mathcal{L} be a very ample invertible \mathcal{O}_X -module. We set $E = \Gamma(X, \mathcal{L})$ and let $X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee)$ be the closed immersion as in the previous section. The Radon transform $\mathcal{R}_{\mathcal{L}j!}\mathcal{F}$ is defined to be $Rp_*q^*j!\mathcal{F}$ using the universal family of hyperplane sections

$$(3.13) \quad X \xleftarrow{q} X \times_{\mathbf{P}} \mathbf{H} \xrightarrow{p} \mathbf{P}^\vee = \mathbf{P}(E).$$

We study the ramification of the cohomology sheaves $\mathcal{R}_{\mathcal{L}j!}^s\mathcal{F} = R^s p_* q^* j!\mathcal{F}$. We define several closed subsets of \mathbf{P}^\vee . Let $D_i, i \in I$ be the irreducible components of dimension 1

of D . For each $i \in I$, let $D_i^\circ \subset D_i$ be a dense open smooth subscheme not meeting $D_{i'}$ for $i' \neq i$ along which the ramification of \mathcal{F} is non-degenerate. We define a finite set Σ of closed points of D by

$$(3.14) \quad \Sigma = D - \bigcup_{i \in I} D_i^\circ.$$

For an irreducible component $D_i, i \in I$ of codimension 1, let $T_{ij}^\circ, j \in J_i$ be the irreducible components of the singular support $SS(j_! \mathcal{F}) \subset T^*(X - \Sigma)$ dominating D_i° and $T_{ij} \subset T^*X$ be the closure. They are irreducible linear closed subschemes of T^*X of dimension 2. We set

$$(3.15) \quad J = \coprod_{i \in I} J_i$$

and let $ij \in J$ denote $j \in J_i \subset J$.

Applying the construction of $P(T)$ (3.1) for linear closed subscheme $T \subset T^*X$, we define closed subvarieties $P(T_X^*X), P(T_{ij})$ for $ij \in J$ and $P(T_x^*X)$ for $x \in \Sigma$ of $X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$. They are irreducible subschemes of $X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ of codimension 2. We define a closed subset $P(j_! \mathcal{F}) \subset X \times_{\mathbf{P}} \mathbf{H}$ to be the union

$$(3.16) \quad P(j_! \mathcal{F}) = P(T_X^*X) \cup \bigcup_{ij \in J} P(T_{ij}) \cup \bigcup_{x \in \Sigma} P(T_x^*X).$$

The image of $P(T_X^*X)$ by the projection $p: X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ is the dual variety X^\vee . Let $T_{ij}^\vee \subset \mathbf{P}^\vee$ denote the image $p(P(T_{ij}))$ for $ij \in J$. The image $H_x = p(P(T_x^*X)) \subset \mathbf{P}^\vee$ is the dual hyperplane $\mathbf{P}(T_x^*\mathbf{P}) = \{H \mid x \in H\}$ for $x \in \Sigma$. Since $P(T_X^*X), P(T_{ij}), P(T_x^*X) \subset X \times_{\mathbf{P}} \mathbf{H}$ are of codimension 2 and $\dim X \times_{\mathbf{P}} \mathbf{H} = \dim \mathbf{P} - 1 + 2 = \dim \mathbf{P}^\vee + 1$, their images in \mathbf{P}^\vee are of codimension ≥ 1 . For $x \in \Sigma$, the canonical morphism $P(T_x^*X) \rightarrow H_x$ is an isomorphism.

Lemma 3.6. *Let \mathcal{L} be an ample invertible \mathcal{O}_X -module. Then, there exists an integer m such that for every $n \geq m$, the invertible \mathcal{O}_X -module $\mathcal{L}^{\otimes n}$ satisfies the condition (L) in Lemma 3.1 and the following condition:*

(R) *The closed subset X^\vee and $T_{ij}^\vee \subset \mathbf{P}^\vee$ for $ij \in J$ are of codimension 1.*

*Further, X^\vee, T_{ij}^\vee for $ij \in J$ and H_x for $x \in \Sigma$ are distinct to each other and the morphisms $P(T^*X) \rightarrow X^\vee$ and $P(T_{ij}) \rightarrow T_{ij}^\vee$ for $ij \in J$ are generically radicial.*

Proof. By Lemma 3.1 and Proposition 3.4, the existence of m as in Lemma 3.6 follows. If the conditions (L) and (R) are satisfied, the remaining assertions follow also from Lemma 3.1. \square

We define a closed subset $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) \subset \mathbf{P}^\vee$ to be the union

$$(3.17) \quad D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) = X^\vee \cup \bigcup_{ij \in J} T_{ij}^\vee \cup \bigcup_{x \in \Sigma} H_x.$$

For an irreducible component D_i of D of dimension 1, the \mathcal{O}_{D_i} -module $\mathcal{L}_i = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i}$ is very ample and the linear subspace $\mathbf{P}_i = \mathbf{P}(E_i) \subset \mathbf{P} = \mathbf{P}(E)$ associated to $E_i = \text{Ker}(E \rightarrow \Gamma(D_i, \mathcal{L}_i))$ is of codimension ≥ 2 .

Lemma 3.7. *The cohomology sheaf $\mathcal{R}_{\mathcal{L}}^s j_! \mathcal{F} = R^s p_* q^* j_! \mathcal{F}$ is 0 except for $s = 0, 1, 2$. The restrictions of $\mathcal{R}_{\mathcal{L}}^s j_! \mathcal{F}$ on the complement $V = \mathbf{P}^\vee - (D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) \cup \bigcup_{i \in I} \mathbf{P}_i)$ is locally constant for every s .*

Proof. Since a hyperplane $H \subset \mathbf{P}$ is defined by a non-zero section $l \in \Gamma(X, \mathcal{L})$, the intersection $X \cap H$ is a Cartier divisor of X . Hence $X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ is flat of relative dimension 1 and $R^s p_* q^* j_! \mathcal{F} = 0$ except for $s = 0, 1, 2$.

Outside $X^\vee \subset \mathbf{P}^\vee$, the proper flat morphism $X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ is smooth. Outside the union $\bigcup_{i \in I} \mathbf{P}_i \cup \bigcup_{x \in \Sigma} H_x \subset \mathbf{P}^\vee$, the closed subscheme $D \times_{\mathbf{P}} \mathbf{H}$ of $X \times_{\mathbf{P}} \mathbf{H}$ is a Cartier divisor flat over \mathbf{P}^\vee . By the definition of T_{ij}° for $ij \in J$, the restriction of $X \times_{\mathbf{P}} \mathbf{H} \rightarrow \mathbf{P}^\vee$ on the open subscheme V is non-characteristic with respect to $j_! \mathcal{F}$. Hence, it is locally acyclic relatively to $j_! \mathcal{F}$ by [24, Proposition 3.15]. Thus $R^s p_* q^* j_! \mathcal{F}$ is locally constant on V for every s by [15, 2.4]. \square

We define the characteristic cycle of $j_! \mathcal{F}$ as a cycle in the cotangent bundle T^*X using the ramification of the Radon transform.

Definition 3.8. *Let \mathcal{L} be a very ample invertible \mathcal{O}_X -module satisfying the conditions (L) in Lemma 3.1 and (R) in Lemma 3.6 and let*

$$(3.18) \quad a(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) = a_X^{\mathcal{L}}(j_! \mathcal{F}) \cdot X^\vee + \sum_{ij \in J} a_{ij}^{\mathcal{L}}(j_! \mathcal{F}) \cdot T_{ij}^\vee + \sum_{x \in \Sigma} a_x^{\mathcal{L}}(j_! \mathcal{F}) \cdot H_x$$

denote the Artin divisor (1.8) of the Radon transform $\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}$. We define the characteristic cycle of $j_! \mathcal{F}$ relative to \mathcal{L} by

$$(3.19) \quad \text{Char}_{\mathcal{L}}(j_! \mathcal{F}) = - \left(\frac{a_X^{\mathcal{L}}(j_! \mathcal{F})}{[P(T_X^* X) : X^\vee]} \cdot [T_X^* X] + \sum_{ij \in J} \frac{a_{ij}^{\mathcal{L}}(j_! \mathcal{F})}{[P(T_{ij}^* X) : T_{ij}^\vee]} \cdot [T_{ij}^\vee] + \sum_{x \in \Sigma} \frac{a_x^{\mathcal{L}}(j_! \mathcal{F})}{[P(T_x^* X) : H_x]} \cdot [H_x] \right)$$

*as a cycle of dimension 2 in the cotangent bundle T^*X .*

We have

$$(3.20) \quad p_* P(\text{Char}_{\mathcal{L}}(j_! \mathcal{F})) = -a(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})$$

by the definition. We study the coefficients in more detail in Proposition 3.10.

We prove an analogue Theorem 3.16 of the Milnor formula [6] in several steps. In the following, we assume that \mathcal{L} is a very ample invertible \mathcal{O}_X -module satisfying the conditions (L) in Lemma 3.1 and (R) in Lemma 3.6. Let $X^{\vee\circ}$ be a smooth dense open subscheme of X^\vee satisfying the following conditions: The intersections with other components of $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})$ (3.17) and with \mathbf{P}_i for $i \in I$ are empty. The inverse image of $P(T_X^* X) \rightarrow X^\vee$ consists of one point for every point of $X^{\vee\circ}$. The ramification of $(\mathcal{R}_{\mathcal{L}}^s j_! \mathcal{F})|_V$ along $X^{\vee\circ}$ is non-degenerate. The restriction $(\mathcal{R}_{\mathcal{L}}^s j_! \mathcal{F})|_{X^{\vee\circ}}$ is locally constant for $s = 0, 1, 2$.

Similarly, we define smooth dense open subschemes $T_{ij}^{\vee\circ} \subset T_{ij}^\vee$ for $ij \in J$ and $H_x^{\vee\circ} \subset H_x$ for $x \in \Sigma$. Let $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})^\circ$ denote the disjoint union

$$(3.21) \quad D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})^\circ = X^{\vee\circ} \cup \bigcup_{ij \in J} T_{ij}^{\vee\circ} \cup \bigcup_{x \in \Sigma} H_x^{\vee\circ}$$

as in (3.17). It is a dense open subscheme of $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})$ and is smooth of codimension 1 in \mathbf{P}^\vee .

Lemma 3.9. *Let L be a line in \mathbf{P}^\vee such that the axis A_L meets X transversely. Let y be a closed point of L corresponding to a hyperplane $H \subset \mathbf{P}$ and suppose that L meets $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})$ at y properly.*

1. *Let $z \in X$ be a closed point not contained in A_L satisfying $y = p_L(z)$. Assume that $p_L: X_L \rightarrow L$ is non-characteristic with respect to (the pull-back of) $j_!\mathcal{F}$ on a neighborhood of $p_L^{-1}(z)$ except at z . Then, we have*

$$(3.22) \quad -(a(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F}), L)_y = (\text{Char}_{\mathcal{L}}(j_!\mathcal{F}), [dp_L])_z.$$

2. *Assume that L meets $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})^\circ$ transversely at y and that the immersion $L \rightarrow \mathbf{P}^\vee$ is non-characteristic with respect to $(\mathcal{R}_{\mathcal{L}}^s j_!\mathcal{F})|_V$ at y for $s = 0, 1, 2$. Then, the intersection $((X \cap H) \times \{y\}) \cap P(j_!\mathcal{F}) \subset X \times_{\mathbf{P}} \mathbf{H}$ consists of one point z and we have*

$$(3.23) \quad \dim \text{tot} \phi_z(j_!\mathcal{F}, p_L) = (a(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F}), L)_y.$$

Proof. 1. By $p_*P(\text{Char}_{\mathcal{L}}(j_!\mathcal{F})) = -a(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})$ (3.20) and Lemma 3.5, we obtain (3.22)

2. Let $\bar{\eta}_y$ denote a geometric generic point of the strict localization of L at y . Then, the distinguished triangle of vanishing cycles gives a distinguished triangle

$$(3.24) \quad \rightarrow (\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})_y \rightarrow (\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})_{\bar{\eta}_y} \rightarrow \phi_z(j_!\mathcal{F}, p_L) \rightarrow .$$

Hence, we have $\dim \text{tot} \phi_z(j_!\mathcal{F}, p_L) = a_y((\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})|_L)$.

By the assumption, the immersion $L \rightarrow \mathbf{P}^\vee$ is non-characteristic at y and the restrictions of cohomology sheaves of $\mathcal{R}_{\mathcal{L}}j_!\mathcal{F}$ are locally constant on $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})^\circ$. Hence, we have $a_y((\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})|_L) = (a(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F}), L)_y$. \square

Proposition 3.10 (cf. [9, p.17 Question]). *The coefficients of $[T_X^*X]$ in $\text{Char}_{\mathcal{L}}(j_!\mathcal{F})$ is the rank of \mathcal{F} . The coefficient of $[T_{ij}]$ for $ij \in J$ is a rational number at least 0 and its denominator is a power of p . The coefficient of $[T_x^*X]$ for $x \in \Sigma$ is an integer at least 0, if x is not an isolated point of D . If $x \in \Sigma$ is an isolated point of D , it is $-\text{rank } \mathcal{F}$.*

Proof. Let L be a line as in Lemma 3.9.2 and y be a point of intersection $L \cap D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})^\circ$. By Lemma 3.9.2, we have $\dim \text{tot} \phi_z(j_!\mathcal{F}, p_L) = (a(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F}), L)_y$. If z is not an isolated point of D , we have $\phi_z^q(j_!\mathcal{F}, p_L) = 0$ except for $q \neq 1$ and the coefficient of the component containing y in $a(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})$ are integers at most 0. Hence, the coefficients in $\text{Char}_{\mathcal{L}}(j_!\mathcal{F})$ are rational numbers at least 0 except for the coefficients of $[T_x^*X]$ for an isolated point $x \in \Sigma$ of D . If z is an isolated point of D , we have $\phi_z^0(j_!\mathcal{F}, p_L) = (j_*\mathcal{F})_z$ and $\phi_z^q(j_!\mathcal{F}, p_L) = 0$ except for $q \neq 0$. Hence, the coefficient of $[T_z^*X]$ is $-\text{rank } \mathcal{F}$.

Assume that y is in $X^{\vee\circ}$. Since \mathcal{F} is locally constant on a neighborhood of u , by the Milnor formula [6], we have $-\dim \text{tot} \phi_z(j_!\mathcal{F}, p_L) = -\text{rank } \mathcal{F} \cdot \dim \text{tot} \phi_z(j_!\Lambda_U, p_L) = \text{rank } \mathcal{F} \cdot (T_X^*X, [dp_L])_{T^*X, z}$. Hence the coefficient of $[T_X^*X]$ is $\text{rank } \mathcal{F}$ by Lemma 3.9..

Since $P(T_{ij}) \rightarrow T_{ij}^\vee$ is generically purely inseparable by Lemma 3.6, the degree $[P(T_{ij}) : T_{ij}^\vee]$ is a power of p . \square

We show the existence of a good pencil for an invertible sheaf satisfying the conditions (L) and (R).

Lemma 3.11. *Let \mathcal{L} be a very ample invertible \mathcal{O}_X -module satisfying the conditions (L) and (R). Then, the open subscheme of the Grassmannian variety \mathbf{G} consisting of lines $L \subset \mathbf{P}^\vee$ satisfying the following conditions (P1)–(P3) is non-empty:*

- (P1) *The axis A_L meets X transversely and does not meet D . The morphism $p_L|_D: D \rightarrow L$ is generically étale.*
- (P2) *The intersection $L \cap D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})$ is contained in $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})^\circ$ and the intersection $L \cap \bigcup_{i \in I} \mathbf{P}_i$ is empty.*
- (P3) *The immersion $L \rightarrow \mathbf{P}^\vee$ is non-characteristic with respect to $j_{V!}(\mathcal{R}_{\mathcal{L}}^s j_!\mathcal{F})_V$ for $s = 0, 1, 2$ for the open immersion $j_V: V \rightarrow \mathbf{P}^\vee$.*

The condition $A_L \cap D = \emptyset$ implies that the intersection $P(T_x^*X) \cap X_L$ consists of only one point for $x \in \Sigma \subset D$. The condition (P1) implies that the inverse image of the intersection $X \cap A_L$ in X_L does not meet the intersections in (P2). The condition (P2) implies that the intersection $Z_L = P(j_!\mathcal{F}) \cap X_L$ consists of finitely many closed points and the restriction $p_L|_{Z_L}: Z_L \rightarrow L$ is an injection.

Proof. Since each condition is an open condition on \mathbf{G} , it suffices to show that there exists a line $L \subset \mathbf{P}^\vee$ satisfying each condition (P1)–(P3), separately.

By Bertini's theorem, there exists a hyperplane $H \in \mathbf{P}^\vee$ meeting X transversely and another hyperplane $H' \in \mathbf{P}^\vee$ meeting $X \cap H$ transversely. Then, for the line $L \subset \mathbf{P}^\vee$ spanned by H and H' , the axis $A_L = H \cap H' \subset \mathbf{P}$ meets X transversely. Similarly, there exists a hyperplane $H \in \mathbf{P}^\vee$ meeting D transversely and another hyperplane $H' \in \mathbf{P}^\vee$ not meeting $D \cap H$. For the line $L \subset \mathbf{P}^\vee$ spanned by H and H' , the intersection $A_L \cap D$ is empty. A line L satisfying these conditions satisfies (P1).

Since $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})^\circ$ is a dense open subscheme of a divisor $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})$ of \mathbf{P} , there exists a line $L \subset \mathbf{P}^\vee$ such that the intersection with $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F}) - D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})^\circ$ is empty. Since \mathbf{P}_i is of codimension ≥ 2 for every $i \in I$, there exists a line $L \subset \mathbf{P}^\vee$ such that the intersection $L \cap \mathbf{P}_i$ is empty for every $i \in I$. A line L satisfying these conditions satisfies (P2).

For $ij \in J$, let $\Sigma_{ij}^\circ \subset \mathbf{D} \times_{\mathbf{P}^\vee} T_{ij}^{\vee\circ}$ be the subset consisting of pairs (L, H) of a hyperplane $H \in T_{ij}^{\vee\circ} \subset \mathbf{P}^\vee$ and a line $L \subset \mathbf{P}^\vee$ passing through it such that the immersion $L \rightarrow \mathbf{P}^\vee$ is *not* non-characteristic at H . Since the closure $\Sigma_{ij} \subset \mathbf{D}$ of Σ_{ij}° is of codimension 2 and since \mathbf{D} is a \mathbf{P}^1 -bundle over \mathbf{G} , its image $Q_{ij} \subset \mathbf{G}$ is of codimension ≥ 1 . We define $\Sigma_X \subset \mathbf{D}$ and $\Sigma_x \subset \mathbf{D}$ for $x \in \Sigma$ similarly. By the same argument, their images $Q_X \subset \mathbf{G}$ and $Q_x \subset \mathbf{G}$ for $x \in \Sigma$ are of codimension at least 1. A line $L \in \mathbf{G}$ not contained in the union of Q_X, Q_{ij} and $Q_x \subset \mathbf{G}$ for $x \in \Sigma$ satisfies (P3). \square

Proposition 3.12. *Let \mathcal{L} be a very ample invertible \mathcal{O}_X -module satisfying the conditions (L) and (R). Let $L \subset \mathbf{P}^\vee$ be a line such that the axis A_L meets X transversely and does not meet D and set $Z = X_L \cap P(j_!\mathcal{F}) \subset X_L$.*

Let u be a closed point of $X - (X \cap A_L)$ satisfying the following condition:

- (u) *$v = p_L(u)$ is an isolated point of $p_L(Z)$ and that u is the unique point in the intersection $Z \cap p_L^{-1}(v)$.*

Then, we have

$$(3.25) \quad -\dim \operatorname{tot} \phi_u(j_!\mathcal{F}, p_L) = (\operatorname{Char}_{\mathcal{L}}(j_!\mathcal{F}), [dp_L])_{T^*X, u}.$$

Proof. By Lemma 3.11, the open subscheme $V \subset \mathbf{G}$ consisting of lines satisfying the conditions (P1)–(P3) is non-empty. We take a line C in \mathbf{G} passing the point $s \in \mathbf{G}$ defined

by L and meeting V . By replacing C by a neighborhood C of s , we may assume that for every point $t \in C - \{s\}$, the corresponding line L_t satisfies the conditions (P1)–(P3).

We applying Proposition 2.5 to the cartesian diagram

$$\begin{array}{ccccccc}
X_L & \longrightarrow & X_C & \longrightarrow & X_G & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} \\
p_L \downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \longrightarrow & L_C & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{P}^\vee \\
\downarrow & & \downarrow & & \downarrow & & \\
s & \longrightarrow & C & \longrightarrow & \mathbf{G} & &
\end{array}$$

and to the pull-back A of the Artin divisor $a(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})$ to $Y = L_C$. We show that the pull-back of $j_! \mathcal{F}$ to X_C is locally acyclic relatively to $X_C \rightarrow C$. By the assumption (P1), the axis A_{L_C} meets $X \times C$ transversely and does not meet $D \times C$. Hence the pull-back of $j_! \mathcal{F}$ is locally constant on a smooth scheme $X_C - (D \times C)$ over C and is locally acyclic relatively to $X_C - (D \times C) \rightarrow C$ by the local acyclicity of smooth morphism. Further, on a neighborhood of $D \times C$, it is the pull-back by the projection and is also locally acyclic relatively to $X_C \rightarrow C$ by [7, Théorème 2.13]. Hence the pull-back of $j_! \mathcal{F}$ to X_C is locally acyclic relatively to $X_C \rightarrow C$.

The pull-back of $j_! \mathcal{F}$ is universally locally acyclic relatively to $X_C \rightarrow L_C$ outside the inverse images of $P(j_! \mathcal{F})$ by [24, Proposition 3.15]. Hence by Lemma 3.9.2 and Proposition 2.5, we obtain $\dim \operatorname{tot} \phi_u(j_! \mathcal{F}, p_L) = (a(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}), L)_v$. Hence (3.25) follows from (3.22). \square

Proposition 3.13. *Let*

$$C \xleftarrow{f} X' \xrightarrow{\varphi} X$$

be an étale morphism $\varphi: X' \rightarrow X$ of smooth surfaces over k and a flat morphism $f: X' \rightarrow C$ to a smooth curve C over k . Assume that X is projective and let \mathcal{F} be a locally constant constructible sheaf on the complement $U = X - D$ of a reduced closed subscheme $D \subsetneq X$. Let u be a closed point of X' such that u is an isolated characteristic point of $f: X' \rightarrow C$ with respect to $j_! \mathcal{F}$ and assume that the restriction $f|_{D'}: D' = D \times_X X' \rightarrow C$ is étale on a neighborhood of u except at u .

Then, for an ample invertible \mathcal{O}_X -module \mathcal{L} , there exists an integer m such that for every integer $n \geq m$, the invertible \mathcal{O}_X -module $\mathcal{L}^{\otimes n}$ is very ample and satisfies the conditions (L) and (R) and we have

$$(3.26) \quad -\dim \operatorname{tot} \phi_u(j'_! \mathcal{F}', f) = ((T^* \varphi)^* \operatorname{Char}_{\mathcal{L}^{\otimes n}}(j_! \mathcal{F}), [df])_{T^* X', u}.$$

Proof. We prove Proposition by reducing to Proposition 3.12 using the stability of nearby cycles Theorem 2.13. By taking an étale morphism $C \rightarrow \mathbf{P}^1$ on a neighborhood of $v = f(c)$, we may assume $C = \mathbf{P}^1$. By Theorem 2.13, there exists an integer $N \geq 1$ such that for a morphism $g: X' \rightarrow C$ congruent to $f \bmod \mathfrak{m}_u^N$, we have an isomorphism $\phi_u(j_! \mathcal{F}, f) \simeq \phi_u(j_! \mathcal{F}, g)$.

Similarly as Proposition 2.1.1, there exists an integer $N \geq 1$ such that for a morphism $g: X' \rightarrow C$ congruent to $f \bmod \mathfrak{m}_u^N$, we have an equality $(T, [dp_L])_{T^* X', u} = (T, [dp_L])_{T^* X', u}$ for every irreducible component T of the singular support $SS(j_! \mathcal{F})$.

By Lemmas 3.1.2 and 3.3.2, there exists an integer $m \geq 1$ such that for every integer $n \geq m$, the invertible $\mathcal{O}_{\bar{X}}$ -module $\mathcal{L}^{\otimes n}$ is very ample and satisfies the conditions (L) and

(N) for the integer $N \geq 1$ above. We show the equality (3.26) for $n \geq m$ above. By changing the notation, we write \mathcal{L} for $\mathcal{L}^{\otimes n}$. We show the existence of a pencil L such that p_L satisfies the conditions in Proposition 3.12 and that the composition $g = p_L \circ \varphi$ is congruent to $f \bmod \mathfrak{m}_u^N$.

Take a hyperplane $H_\infty \in \mathbf{P}^\vee$ not contained in the union $H_u \cup D(\mathcal{R}_{\mathcal{L}j_i}\mathcal{F})$ and a section $l_\infty \in E = \Gamma(X, \mathcal{L})$ defining H_∞ . Then, u is not contained in H_∞ , the hyperplane section $X_\infty = X \cap H_\infty$ is smooth and $D_\infty = D \cap H_\infty$ is étale and does not contain $x \in \Sigma$. We apply Lemma 3.3 to the family of subschemes T_X^*X, T_{ij} for $ij \in J$, T_x^*X for $x \in \Sigma \cup D_\infty$ and $T_{X_\infty}^*X$. Then, there exists $l \in E_f$ satisfying the conditions loc. cit. for this family. By the rational function l/l_∞ , we also identify $L = \mathbf{P}^1$.

We set $X_0^\circ = X_0 - \{u\}$. The condition that the intersection $T_{X_0^\circ}^*X \cap T_{X_\infty}^*X$ is contained in the 0-section means that X_∞ and X_0° meet transversely and hence the axis A_L of the pencil L spanned by l and l_∞ meets X transversely. The condition that the intersection $T_{X_0^\circ}^*X \cap T_x^*X$ is contained in the 0-section for $x \in D_\infty$ means that the axis A_L does not meet D .

Let $p_L: X_L \rightarrow L$ be the morphism defined by the pencil L and set $v = p_L(u) \in L$. By the conditions that H_∞ is not contained in $D(\mathcal{R}_{\mathcal{L}j_i}\mathcal{F})$ and that $X \cap A_L$ is contained in U , the morphism $p_L: X_L \rightarrow L$ is smooth and is non-characteristic with respect to the pull-back of $j_i\mathcal{F}$ except at finitely many closed points. The condition that the intersection $T_{X_0^\circ}^*X \cap T_X^*X$ is contained in the 0-section means that the morphism $p_L: X_L \rightarrow L$ is smooth on a neighborhood of $p_L^{-1}(v) - \{u\}$. Let $Z_L \subset X_L$ denote the closed subset outside of which the morphism $p_L: X_L \rightarrow L$ is non-characteristic with respect to $j_i\mathcal{F}$. Then, the condition that the intersection $T_{X_0^\circ}^*X \cap T_{ij}$ is contained in the 0-section means that the immersion $X_0^\circ \rightarrow X$ is non-characteristic with respect to $j_i\mathcal{F}$ and that the intersection $X_0^\circ \cap Z_L$ is empty. Thus, the condition (u) in Proposition 3.12 is satisfied and we have an equality $-\dim \text{tot}_{\varphi(u)}(j_i\mathcal{F}, p_L) = (\text{Char}_{\mathcal{L}}(j_i\mathcal{F}), [dp_L])_{T^*X, \varphi(u)}$.

The congruence $l/l_\infty \equiv f \bmod \mathfrak{m}_u^N$ means that the composition $p_L \circ \varphi: X \rightarrow L$ is congruent to $f: X \rightarrow C \bmod \mathfrak{m}_u^N$. Thus, by Theorem 2.13, we have an isomorphism $\phi_u(\varphi^*j_i\mathcal{F}, f) \rightarrow \phi_{\varphi(u)}(j_i\mathcal{F}, p_L)$. Since we also have $((T^*\varphi)^*\text{Char}_{\mathcal{L}}(j_i\mathcal{F}), [df])_{T^*X, u} = (\text{Char}_{\mathcal{L}}(j_i\mathcal{F}), [dp_L])_{T^*X', \varphi(u)}$, the assertion follows. \square

Corollary 3.14. *Let $f: X' \rightarrow X$ be an étale morphism of smooth surfaces over k . Let $\bar{X} \supset X$ and $\bar{X}' \supset X'$ be projective smooth surfaces containing X and X' as dense open subschemes and let \mathcal{L} and \mathcal{L}' be a very ample invertible \mathcal{O}_X -module and $\mathcal{O}_{X'}$ -module satisfying the conditions (L) and (R).*

Let U be a dense open subscheme of X and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on U . Let $\bar{j}: U \rightarrow \bar{X}$ and $\bar{j}': U' = U \times_X X' \rightarrow \bar{X}'$ be the open immersions and let \mathcal{F}' be the pull-back of \mathcal{F} on U' . Then, we have

$$(T^*f)^*(\text{Char}_{\mathcal{L}}(\bar{j}_i\mathcal{F})|_X) = \text{Char}_{\mathcal{L}'}(\bar{j}'_i\mathcal{F}')|_{X'}.$$

Proof. Let $j: U \rightarrow X$ and $j': U' \rightarrow X'$ be the open immersions. Let $u' \in X'$ and $u = \varphi(u')$ be closed points let $p_L: \bar{X}_K \rightarrow L$ and $p_{L'}: \bar{X}'_{L'} \rightarrow L'$ be morphisms satisfying the conditions (P1)–(P3) for u and u' respectively. Then, by Proposition 3.12, we have

$$\begin{aligned} -\dim \text{tot}_u \phi(p_L, j_i\mathcal{F}) &= (\text{Char}_{\mathcal{L}}(\bar{j}_i\mathcal{F}), [dp_L])_{T^*X, u}, \\ -\dim \text{tot}_{u'} \phi(p_{L'}, j'_i\mathcal{F}') &= (\text{Char}_{\mathcal{L}'}(\bar{j}'_i\mathcal{F}'), [dp_{L'}])_{T^*X', u'}. \end{aligned}$$

By Proposition 3.13, there exists an integer n such that invertible $\mathcal{O}_{\bar{X}}$ -module $\mathcal{M} = \mathcal{L}^{\otimes n}$ is very ample, satisfying the condition (L) and (R) and

$$\begin{aligned} -\dim \operatorname{tot}_u \phi(p_L, j_! \mathcal{F}) &= (\operatorname{Char}_{\mathcal{M}}(\bar{j}_! \mathcal{F}), [dp_L])_{T^*X, u}, \\ -\dim \operatorname{tot}_{u'} \phi(p_{L'}, j'_! \mathcal{F}') &= ((T^*f)^* \operatorname{Char}_{\mathcal{M}}(\bar{j}_! \mathcal{F}), [dp_{L'}])_{T^*X', u'}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} (\operatorname{Char}_{\mathcal{L}}(\bar{j}_! \mathcal{F}), [dp_L])_{T^*X, u} &= (\operatorname{Char}_{\mathcal{M}}(\bar{j}_! \mathcal{F}), [dp_L])_{T^*X, u}, \\ (\operatorname{Char}_{\mathcal{L}'}(\bar{j}_! \mathcal{F}), [dp_{L'}])_{T^*X', u} &= ((T^*f)^* \operatorname{Char}_{\mathcal{M}}(\bar{j}_! \mathcal{F}), [dp_{L'}])_{T^*X', u'}. \end{aligned}$$

Letting u, L and L' vary, this implies that the coefficients in $(T^*f)^* \operatorname{Char}_{\mathcal{L}}(j_! \mathcal{F})$ and $\operatorname{Char}_{\mathcal{L}'}(\bar{j}_! \mathcal{F})$ are both equal to the corresponding coefficients in $(T^*f)^* \operatorname{Char}_{\mathcal{M}}(\bar{j}_! \mathcal{F})$. \square

Corollary 3.14 means that the characteristic cycle $\operatorname{Char}_{\mathcal{L}}(j_! \mathcal{F})$ is independent of the choice of a very ample \mathcal{O}_X -module \mathcal{L} satisfying (L) and (R) and that the construction of $\operatorname{Char}_{\mathcal{L}}(j_! \mathcal{F})$ is étale local. Thus, we can make the following definition.

Definition 3.15. *Let X be a smooth surface over k and U be the complement of a Cartier divisor. Let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on U . Then, we define $\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F})$ to be the restriction to T^*X of $\operatorname{Char}_{\mathcal{L}}(\bar{j}_! \mathcal{F})$ for a smooth compactification $X \rightarrow \bar{X}$, the composition $\bar{j}: U \rightarrow \bar{X}$ and a very ample invertible $\mathcal{O}_{\bar{X}}$ -module \mathcal{L} satisfying (L) and (R).*

The construction of $\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F})$ is additive in the sense that we have

$$\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}) = \operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}') + \operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}'')$$

for an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of locally constant constructible sheaves on $U = X - D$. We record the equality (3.25) for the convenience of the reference.

Theorem 3.16 (cf. [9, p.7 Principe]). *Let X be a smooth surface over k and let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on a dense open subscheme U . Let $f: X \rightarrow C$ be a flat morphism to a smooth curve and u be a closed point of X . Assume that u is an isolated characteristic point of f with respect to $j_! \mathcal{F}$ and that D is étale over C on a neighborhood of u except at u . Then, we have*

$$(3.27) \quad -\dim \operatorname{tot}_u \phi_u(j_! \mathcal{F}, f) = (\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}), [df])_{T^*X, u}.$$

Proof. Clear from Proposition 3.13. \square

We prove a variant of Theorem 3.16 for a normal surface later at Proposition 3.20.

3.3 Euler characteristic and the characteristic cycle

We compute the Euler characteristic. Let X be a smooth connected surface over a perfect field k , let $D \subsetneq X$ be a reduced closed subscheme and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on $U = X - D$. Let $Y \rightarrow X$ be a closed immersion of a smooth curve such that the immersion $Y \rightarrow X$ is non-characteristic with respect to $j_! \mathcal{F}$. Then let $\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}|_Y)$ denote -1 -times the cycle of T^*Y defined as the image of the fiber $\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}) \times_X Y$ by the surjection $T^*X \times_X Y \rightarrow T^*Y$.

Lemma 3.17. *Let X be a projective smooth connected surface, $D \subset X$ be a reduced closed subscheme and C be a proper smooth connected curve of genus g over an algebraically closed field k . Let $f: X \rightarrow C$ be a proper flat morphism over k such that the restriction $f|_D: D \rightarrow C$ is finite. Let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on $U = X - D$.*

Assume that $f: X \rightarrow C$ is non-characteristic with respect to $j_!\mathcal{F}$ on the complement of a finite set Z of closed points of X . Let $c \in C$ be a closed point such that on a neighborhood $V \subset C$ of c , the morphism $X \times_C V \rightarrow V$ is smooth and non-characteristic with respect to $j_!\mathcal{F}$ and set $Y = X \times_C c$.

1. *We have*

$$(3.28) \quad \chi_c(U, \mathcal{F}) = (2 - 2g) \cdot \chi_c(U \cap Y, \mathcal{F}|_{U \cap Y}) - \sum_{x \in Z} \dim \operatorname{tot} \phi_x(j_!\mathcal{F}, f).$$

2. *We have*

$$(3.29) \quad (\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}), T_X^*X)_{T^*X} = (2 - 2g) \cdot (\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y), T_Y^*Y)_{T^*Y} + \sum_{x \in Z} (\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}), [df])_{T^*X, x}.$$

Proof. 1. By the assumption that $X \times_C V \rightarrow V$ is non-characteristic with respect to $j_!\mathcal{F}$, the cohomology sheaves of $Rf_*j_!\mathcal{F}$ are locally constant on V similarly as in the proof of Lemma 3.7 and we have $\operatorname{rank} (Rf_*j_!\mathcal{F})_V = \chi_c(U \cap Y, \mathcal{F}|_{U \cap Y})$. Hence it suffices to apply the Grothendieck-Ogg-Shafarevich formula [13, Théorème 7.1] to compute $\chi_c(U, \mathcal{F}) = \chi(C, Rf_*j_!\mathcal{F})$.

2. By the cartesian diagram

$$\begin{array}{ccccc} X & \longrightarrow & T^*C \times_C X & \longrightarrow & T^*X \\ f \downarrow & & \downarrow p & & \\ C & \longrightarrow & T^*C & & \end{array}$$

we have

$$(\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}), T_X^*X)_{T^*X} = (p_*A, T_C^*C)_{T^*C}$$

where we set $A = (\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*C \times_C X)_{T^*X}$.

Since $X \times_C V \rightarrow V$ is assumed non-characteristic, the push-forward p_*A is supported in the union of the 0-section and the inverse image of $C - V$. Hence, it is the sum $A_1 + A_2$ of a multiple A_1 of the zero-section and a linear combination A_2 of fibers. We have

$$(A_1, T_C^*C)_{T^*C} = (p_*A, T_C^*C)_{T^*C} \cdot (T_C^*C, T_C^*C)_{T^*C} = (\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*C \times_C Y)_{T^*X} \cdot (2g - 2).$$

By the exact sequence $0 \rightarrow T^*C \times_C Y \rightarrow T^*X \times_X Y \rightarrow T^*Y \rightarrow 0$ and the definition of $\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y)$, we have $(\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*C \times_C Y)_{T^*X} = -(\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y), T_Y^*Y)_{T^*Y}$ and $(A_1, T_C^*C)_{T^*C}$ equals the first term in the right hand side of (3.29). Since $(A_2, T_C^*C)_{T^*C}$ is equal to the second term, the equality (3.29) is proved. \square

Theorem 3.18 (cf. [9, p.13 Corollaire]). *Let X be a projective smooth surface over an algebraically closed field k of characteristic $p > 0$, U be a dense open subscheme and $j: U \rightarrow X$ be the open immersion. Let Λ be a finite field of characteristic $\ell \neq p$ and \mathcal{F} be a locally constant constructible sheaf of Λ -modules on U . Then, we have an equality*

$$(3.30) \quad \chi_c(U, \mathcal{F}) = (\operatorname{Char}^{\mathcal{R}}(j_!\mathcal{F}), T_X^*X)_{T^*X}.$$

Theorem 3.18 is proved in [20, Théorème 1.2.1] under the following “non-feroce” assumption on \mathcal{F} : There exists a finite Galois covering V of U trivializing \mathcal{F} such that for every point $\xi \in X$ of codimension 1, the pull-back $V \times_U \text{Spec } K_\xi$ to the local field $K_\xi = \text{Frac}(\hat{\mathcal{O}}_{X,\xi})$ at ξ is isomorphic to $\coprod \text{Spec } L_i$ for finite extensions L_i of local fields such that the residue fields are separable over that of K_ξ .

Proof. Let \mathcal{L} be a very ample invertible \mathcal{O}_X -module satisfying the conditions (L) and (R). By Lemma 3.11 there exists a line $L \subset \mathbf{P}^\vee$ satisfying the conditions (P1)–(P3). Let H be the hyperplane corresponding to a closed point of L not contained in $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})$ and $Y = X \cap H$ be the hyperplane section. We compare (3.28) and (3.29) for the blow-up X_L and the pull-back \mathcal{F}_L of \mathcal{F} to $U_L = U \times_X X_L$. Then, since the axis A_L meets X transversely and does not meet D , we have

$$\chi_c(U_L, \mathcal{F}) = \chi_c(U, \mathcal{F}) + \text{rank } \mathcal{F} \cdot \deg(X \cap A_L)$$

and

$$(\text{Char}^{\mathcal{R}}(j_! \mathcal{F}_L), T_{X_L}^* X_L)_{T^* X_L} = (\text{Char}^{\mathcal{R}}(j_! \mathcal{F}), T_X^* X)_{T^* X} + \text{rank } \mathcal{F} \cdot \deg(X \cap A_L).$$

By the Grothendieck-Ogg-Shafarevich formula, we have

$$(3.31) \quad \chi_c(U \cap Y, j_! \mathcal{F}|_{U \cap Y}) = (\text{Char}(j_! \mathcal{F}|_Y), T_Y^* Y)_{T^* Y}.$$

Hence, by (3.28), (3.29) and Theorem 3.16, we have

$$(3.32) \quad \chi_c(U, \mathcal{F}) - (\text{Char}^{\mathcal{R}}(j_! \mathcal{F}), T_X^* X)_{T^* X} = 2(\text{Char}(j_! \mathcal{F}|_Y) - \text{Char}^{\mathcal{R}}(j_! \mathcal{F}|_Y), T_Y^* Y)_{T^* Y}.$$

For $ij \in J$, let D_{ij}° be a finite scheme over D_i° such that $T_{ij} \times_{D_i} D_i^\circ$ is a line bundle over D_{ij}° . We put $\text{Char}^{\mathcal{R}}(j_! \mathcal{F}) = \text{rank } \mathcal{F} \cdot [T_X^* X] + \sum_{ij \in J} s_{ij}^{\mathcal{R}}(j_! \mathcal{F})[T_{ij}] + \sum_{x \in \Sigma} s_x^{\mathcal{R}}(j_! \mathcal{F})[T_x^* X]$ and define an effective Cartier divisor $DT^{\mathcal{R}}(j_! \mathcal{F})$ supported on D by

$$(3.33) \quad DT^{\mathcal{R}}(j_! \mathcal{F}) = \sum_{ij \in J} s_{ij}^{\mathcal{R}}(j_! \mathcal{F}) \cdot [D_{ij}^\circ : D_i^\circ] \cdot D_i.$$

Since the coefficients of the 0-section $T_Y^* Y$ in $\text{Char}(j_! \mathcal{F}|_Y)$ and $\text{Char}^{\mathcal{R}}(j_! \mathcal{F}|_Y)$ are both $\text{rank } \mathcal{F}$ and the other coefficients are defined by the intersection $(DT(j_! \mathcal{F}), Y)_X$ and $(DT^{\mathcal{R}}(j_! \mathcal{F}), Y)_X$, the right hand side of (3.32) is equal to $2(DT(j_! \mathcal{F}) - DT^{\mathcal{R}}(j_! \mathcal{F}), c_1(\mathcal{L}))_X$. Namely, we have

$$(3.34) \quad \chi_c(U, \mathcal{F}) - (\text{Char}^{\mathcal{R}}(j_! \mathcal{F}), T_X^* X)_{T^* X} = 2(DT(j_! \mathcal{F}) - DT^{\mathcal{R}}(j_! \mathcal{F}), c_1(\mathcal{L}))_X.$$

The left hand side of (3.34) is independent of the choice of an ample invertible \mathcal{O}_X -module \mathcal{L} satisfying the conditions (L) and (R). Since the Néron-Severi group is generated by the classes of ample invertible sheaves, the difference $DT(j_! \mathcal{F}) - DT^{\mathcal{R}}(j_! \mathcal{F})$ is a divisor numerically equivalent to 0 by Lemma 3.1.2. Hence the right hand side of (3.34) is 0 and we obtain (3.30). \square

Proposition 3.19. *The restriction of $\text{Char}^{\mathcal{R}}(j_! \mathcal{F})$ to the non-degenerate locus is equal to $\text{Char}(j_! \mathcal{F})$ whose definition is recalled in Section 1.*

Proof. It suffices to show that the coefficients of T_X^*X, T_{ij} for $ij \in J$ and T_x^*X for $x \in \Sigma$ in $\text{Char}^{\mathcal{R}}(j_!\mathcal{F})$ are equal to the corresponding ones in $\text{Char}(j_!\mathcal{F})$ as long as the latter is defined. By Proposition 3.10, the coefficient of T_X^*X in $\text{Char}^{\mathcal{R}}(j_!\mathcal{F})$ is $\text{rank } \mathcal{F}$ and the assertion follows in this case.

We deduce the assertion on the coefficients of T_{ij} for $ij \in J$ from that $DT(j_!\mathcal{F}) - DT^{\mathcal{R}}(j_!\mathcal{F})$ is numerically equivalent to 0 proved at the end of the proof of Theorem 3.18. Let D_1 be an irreducible component of dimension 1 of D . The assertion is étale local by Corollary 3.14. By the additivity of the characteristic cycles, we may assume that J_1 consists of one element 1. Let $s_{11}(j_!\mathcal{F})$ and $s_{11}^{\mathcal{R}}(j_!\mathcal{F})$ be the coefficients of T_{11} in $\text{Char}(j_!\mathcal{F})$ and in $\text{Char}^{\mathcal{R}}(j_!\mathcal{F})$.

By approximation, there exists a finite Galois extension L of the function field K of Galois group G of X such that the local field K_1 splits completely and that the inertia group at K_i for $i \in I, i \neq 1$ acts trivially on the stalk of \mathcal{F} . Let $Y \rightarrow X$ be the normalization in L and let $X' \rightarrow Y$ be a resolution of singularities.

Let H be the class of an ample line bundle on Y and let $D_{1,Y}$ be the inverse image of D_1 in Y . Then, since the divisor $DT(f^*j_!\mathcal{F}) - DT^{\mathcal{R}}(f^*j_!\mathcal{F})$ of X' is numerically equivalent to 0, we have $(DT(f^*\mathcal{F}) - DT^{\mathcal{R}}(f^*\mathcal{F}), H)_Y = 0$. Since the right hand side is $(s_{11}(j_!\mathcal{F}) - s_{11}^{\mathcal{R}}(j_!\mathcal{F})) \cdot [D_{11}^\circ : D_1^\circ] \times (D_{1,Y}, H)_Y$ and $(D_{1,Y}, H)_Y > 0$, we have $s_{11}(j_!\mathcal{F}) = s_{11}^{\mathcal{R}}(j_!\mathcal{F})$. Thus the assertion for the coefficient of T_{ij} for $ij \in J$ is proved.

Assume that D has simple normal crossing and that \mathcal{F} is non-degenerate along D and let u be a closed point of D . We show that the coefficients of T_u^*X in $\text{Char}(j_!\mathcal{F})$ and $\text{Char}^{\mathcal{R}}(j_!\mathcal{F})$ are equal. It suffices to consider the cases where \mathcal{F} is tame ramified along D and is totally wild ramified separately.

Assume that \mathcal{F} is tamely ramified along D . If u is a smooth point of D , let $f: X \rightarrow C$ be a morphism to a smooth curve defined on a neighborhood of u such that the restriction $f|_D: D \rightarrow C$ is étale. Then, by [15], $f: X \rightarrow C$ is locally acyclic relatively to $j_!\mathcal{F}$ and we have $\phi_u(j_!\mathcal{F}, f) = 0$. Hence, by Theorem 3.16, we have $(\text{Char}^{\mathcal{R}}(j_!\mathcal{F}), [df]) = 0$ and the coefficient of T_u^*X in $\text{Char}^{\mathcal{R}}(j_!\mathcal{F})$ is zero. Thus the assertion follows in this case.

Assume that u is in the intersection of two components D_1 and D_2 of D . Since the local tame monodromy is abelian, we may assume that \mathcal{F} is of rank 1. Let $f: X \rightarrow C$ be a morphism to a smooth curve defined on a neighborhood of u such that the restriction $f|_{D_1}: D_1 \rightarrow C$ and $f|_{D_2}: D_2 \rightarrow C$ are étale. Then, we have $\psi_u^q(j_!\mathcal{F}, f) = 0$ for $q \neq 1$ and we have $\dim \psi_u^1(j_!\mathcal{F}, f) = 1$ by [19]. Let $\pi: X' \rightarrow X$ be the blow-up at u and set $v = f(u)$. Let E be the exceptional divisor and let $w_1, w_2, w_3 \in E$ be the intersection with the proper transforms of D_1, D_2 and of the fiber $f^{-1}(f(u))$ respectively and set $E^\circ = E - \{w_1, w_2, w_3\}$.

An elementary computation as in [23] shows the following: $\psi^0(\pi^*j_!\mathcal{F}, f)|_{E^\circ}$ is a locally constant constructible sheaf of rank 1 tamely ramified at w_1, w_2, w_3 with a tame Galois action of the local field K_v of C at v and $\psi^q(\pi^*j_!\mathcal{F}, f)|_{E^\circ} = 0$ for $q \neq 0$. We have $\psi_{w_1}(\pi^*j_!\mathcal{F}, f) = \psi_{w_2}(\pi^*j_!\mathcal{F}, f) = 0$. We have $\psi_{w_3}^q(\pi^*j_!\mathcal{F}, f) = 0$ except for $q = 0, 1$ and $\psi_{w_3}^q(\pi^*j_!\mathcal{F}, f)$ for $q = 0, 1$ have the same dimension with a tame Galois action of the local field K_v . Thus, the Galois action of the local field K_v on $\psi(j_!\mathcal{F}, f) = R\Gamma(E, \psi(\pi^*j_!\mathcal{F}, f)|_E)$ is tamely ramified and $\dim \text{tot} \psi_u^1(j_!\mathcal{F}, f) = 1$. Hence, by Theorem 3.16, we have $(\text{Char}^{\mathcal{R}}(j_!\mathcal{F}), [df]) = 1$ and the coefficient of T_u^*X in $\text{Char}^{\mathcal{R}}(j_!\mathcal{F})$ is 1. Thus the assertion also follows in this case.

Assume that \mathcal{F} is totally wildly ramified along D . Let $f: X \rightarrow C$ be a morphism to a smooth curve defined on a neighborhood of u that is non-characteristic with respect to

$j_! \mathcal{F}$. Then, $f: X \rightarrow C$ is locally acyclic relatively to and $j_! \mathcal{F}$ we have $\psi_u(j_! \mathcal{F}, f) = 0$ by [24, Proposition 3.15]. Hence the assertion follows as above. \square

Proposition 3.20. *Let X be a normal surface over k and let \mathcal{F} be a locally constant constructible sheaf of Λ -modules on a dense open subscheme $U = X - D$. Let $f: X \rightarrow C$ be a flat morphism to a smooth curve and u be a closed point of X . Assume that u is an isolated characteristic point of f with respect to $j_! \mathcal{F}$ and that $D - \{u\}$ is étale over C on a neighborhood of u . Let $\pi: X' \rightarrow X$ be a resolution, \mathcal{F}' be the pull-back of \mathcal{F} and $E = \pi^{-1}(u)$ be the inverse image. Then, we have*

$$(3.35) \quad -\dim \operatorname{tot} \phi_u(j_! \mathcal{F}, f) = (\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}'), [df])_{T^*X', E}.$$

Proof. By resolution, we may assume that X is projective and $X - \{u\}$ is smooth over k . Let $N \geq 1$ be an integer such that and $g \equiv f \bmod \mathfrak{m}_u^N$ implies an isomorphism $\phi_u(j_! \mathcal{F}, f) \simeq \phi_u(j_! \mathcal{F}, g)$ by Theorem 2.13. We take an ample invertible \mathcal{O}_X -module \mathcal{L} satisfying the conditions (L) and (R) and take a pencil L such that $p_L: X_L \rightarrow L$ satisfies the condition in Proposition 3.12 and $f \equiv p_L \bmod \mathfrak{m}_u^N$.

Similarly as Lemma 3.17, we obtain equalities

$$(3.36) \quad \begin{aligned} & \chi_c(U', \mathcal{F}) + \operatorname{rank} \mathcal{F} \cdot \deg(X \cap A_L) \\ &= 2\chi_c(U' \times_X Y, \mathcal{F}|_{U' \times_X Y}) - \dim \operatorname{tot} \phi_u(j_! \mathcal{F}, p_L) - \sum_{x \in X} \dim \operatorname{tot} \phi_x(j_! \mathcal{F}, p_L). \end{aligned}$$

and

$$(3.37) \quad \begin{aligned} & (\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}), T_{X'}^* X')_{T^*X'} + \operatorname{rank} \mathcal{F} \cdot \deg(X \cap A_L) \\ &= 2(\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}|_Y), T_Y^* Y)_{T^*Y} + (\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}), [df])_{T^*X', E} + \sum_{x \in X} (\operatorname{Char}^{\mathcal{R}}(j_! \mathcal{F}), [dp_L])_{T^*X, x}. \end{aligned}$$

The corresponding terms in (3.36) and (3.37) are equal to each other except for the second terms in the right hand, by Theorems 3.18 and 3.16 and the Grothendieck-Ogg-Shafarevich formula. Hence we have an equality also for the second terms and the assertion follows. \square

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